

FUNCTORS BETWEEN REEDY MODEL CATEGORIES OF DIAGRAMS

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ABSTRACT. If \mathcal{D} is a Reedy category and \mathcal{M} is a model category, the category $\mathcal{M}^{\mathcal{D}}$ of \mathcal{D} -diagrams in \mathcal{M} is a model category under the Reedy model category structure. If $\mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor between Reedy categories, then there is an induced functor of diagram categories $\mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$. Our main result is a characterization of the Reedy functors $\mathcal{C} \rightarrow \mathcal{D}$ that induce right or left Quillen functors $\mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ for every model category \mathcal{M} . We apply these results to various situations, and in particular show that certain important subdiagrams of a fibrant multicosimplicial object are fibrant.

CONTENTS

1. Introduction	1
2. Reedy model category structures	2
2.1. Reedy categories and their diagram categories	3
2.2. Filtrations of Reedy categories	4
2.3. Reedy functors	5
2.4. Opposites	11
2.5. Quillen functors	15
2.6. Cofinality	16
3. Examples	16
3.1. A Reedy functor that is not fibering	16
3.2. Truncations	17
3.3. Skeleta	18
3.4. (Multi)cosimplicial and (multi)simplicial objects	18
4. Proofs of the main theorems	22
4.1. Proof of Theorem 4.2	22
4.2. Proof of Theorem 4.3	30
4.3. Proof of Theorem 1.2	34
References	34

1. INTRODUCTION

The interesting functors between model categories are the *left Quillen functors* and *right Quillen functors* (see [H1, Def. 8.5.2]). Left Quillen functors preserve cofibrant objects and take weak equivalences between cofibrant objects into weak

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equivalences, while right Quillen functors preserve fibrant objects and take weak equivalences between fibrant objects into weak equivalences (see Proposition 2.33). In this paper, we study Quillen functors between diagram categories with the Reedy model category structure..

In more detail, if \mathcal{C} is a Reedy category and \mathcal{M} is a model category, then there is a *Reedy model category structure* on the category $\mathcal{M}^{\mathcal{C}}$ of \mathcal{C} -diagrams in \mathcal{M} (see Definition 2.3 and Theorem 2.5). The original (and most well known) examples of Reedy model category structures are the model categories of *cosimplicial objects in a model category* and of *simplicial objects in a model category*.

Any functor $G: \mathcal{C} \rightarrow \mathcal{D}$ between Reedy categories induces a functor $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ of diagram categories (see Definition 2.12), and it is important to know when such a functor G^* is a left Quillen functor or a right Quillen functor. The results in this paper provide a complete characterization of the Reedy functors (functors between Reedy categories that preserve the structure; see Definition 2.10) between diagram categories for which this is the case.

In more detail, we introduce the notion of a *Reedy functor* between Reedy categories (see Definition 2.10), which is a functor that preserves the Reedy category structure. We then characterize those Reedy functors that induce right Quillen functors on the diagram categories and those that induce left Quillen functors on the diagram categories. More precisely, we have

Theorem 1.1. *If $G: \mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor, then the induced functor of diagram categories $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ is a right Quillen functor for every model category \mathcal{M} if and only if G is a fibering Reedy functor (see Definition 2.16).*

We also have a dual result:

Theorem 1.2. *If $G: \mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor, then the induced functor of diagram categories $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ is a left Quillen functor for every model category \mathcal{M} if and only if G is a cofibered Reedy functor (see Definition 2.16).*

The structure of the paper is as follows: We provide some background on Reedy categories and functors in Section 2, including discussions of filtrations, opposites, Quillen functors, and cofinality. The only new content for this part is in Section 2.3, where we define inverse and direct \mathcal{C} -factorizations and (co)fibered Reedy functors, and prove some results about them. We then discuss several examples and applications of Theorem 1.1 and Theorem 1.2 in Section 3. More precisely, we look at the subdiagrams given by truncations, diagrams defined as skeleta, and three kinds of subdiagrams determined by (co)simplicial and multi(co)simplicial diagrams: Restricted (co)simplicial objects, diagonals of multi(co)simplicial objects, and slices of multi(co)simplicial objects. We then finally present the proofs of Theorem 1.1 and Theorem 1.2 in Section 4. Theorem 1.1 will follow immediately from Theorem 4.1, which is its slight elaboration. Theorem 1.2 can be proved by dualizing the proof of Theorem 1.1, but we will instead derive it in Section 4.3 from Theorem 1.1 and a careful discussion of opposite categories.

2. REEDY MODEL CATEGORY STRUCTURES

In this section, we give the definitions and results needed for the statements and proofs of our theorems. We assume the reader is familiar with the basic language of model categories. The material here is standard, with the exception of Section 2.3

where the key notions for characterizing Quillen functors between Reedy model categories are introduced (Definition 2.13 and Definition 2.16).

2.1. Reedy categories and their diagram categories.

Definition 2.1. A *Reedy category* is a small category \mathcal{C} together with two subcategories $\vec{\mathcal{C}}$ (the *direct subcategory*) and $\overleftarrow{\mathcal{C}}$ (the *inverse subcategory*), both of which contain all the objects of \mathcal{C} , in which every object can be assigned a nonnegative integer (called its *degree*) such that

- (1) Every non-identity map of $\vec{\mathcal{C}}$ raises degree.
- (2) Every non-identity map of $\overleftarrow{\mathcal{C}}$ lowers degree.
- (3) Every map g in \mathcal{C} has a unique factorization $g = \overrightarrow{g} \overleftarrow{g}$ where \overrightarrow{g} is in $\vec{\mathcal{C}}$ and \overleftarrow{g} is in $\overleftarrow{\mathcal{C}}$.

Remark 2.2. The function that assigns to every object of a Reedy category its degree is not a part of the structure, but we will generally assume that such a degree function has been chosen.

Definition 2.3. Let \mathcal{C} be a Reedy category and let \mathcal{M} be a model category.

- (1) A \mathcal{C} -*diagram* in \mathcal{M} is a functor from \mathcal{C} to \mathcal{M} .
- (2) The category $\mathcal{M}^{\mathcal{C}}$ of \mathcal{C} -diagrams in \mathcal{M} is the category with objects the functors from \mathcal{C} to \mathcal{M} and with morphisms the natural transformations of such functors.

In order to describe the *Reedy model category structure* on the diagram category $\mathcal{M}^{\mathcal{C}}$ in Theorem 2.5, we first define the *latching maps* and *matching maps* of a \mathcal{C} -diagram in \mathcal{M} as follows.

Definition 2.4. Let \mathcal{C} be a Reedy category, let \mathcal{M} be a model category, let \mathbf{X} and \mathbf{Y} be \mathcal{C} -diagrams in \mathcal{M} , let $f: \mathbf{X} \rightarrow \mathbf{Y}$ be a map of diagrams, and let α be an object of \mathcal{C} .

- (1) The *latching category* $\partial(\vec{\mathcal{C}} \downarrow \alpha)$ of \mathcal{C} at α is the full subcategory of $(\vec{\mathcal{C}} \downarrow \alpha)$ (the category of objects of $\vec{\mathcal{C}}$ over α ; see [H1, Def. 11.8.1]) containing all of the objects except the identity map of α .
- (2) The *latching object* of \mathbf{X} at α is

$$L_{\alpha} \mathbf{X} = \operatorname{colim}_{\partial(\vec{\mathcal{C}} \downarrow \alpha)} \mathbf{X}$$

and the *latching map* of \mathbf{X} at α is the natural map

$$L_{\alpha} \mathbf{X} \longrightarrow \mathbf{X}_{\alpha} .$$

We will use $L_{\alpha}^{\mathcal{C}} \mathbf{X}$ to denote the latching object if the indexing category is not obvious.

- (3) The *relative latching map* of $f: \mathbf{X} \rightarrow \mathbf{Y}$ at α is the natural map

$$\mathbf{X}_{\alpha} \amalg_{L_{\alpha} \mathbf{X}} L_{\alpha} \mathbf{Y} \longrightarrow \mathbf{Y}_{\alpha} .$$

- (4) The *matching category* $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ of \mathcal{C} at α is the full subcategory of $(\alpha \downarrow \overleftarrow{\mathcal{C}})$ (the category of objects of $\overleftarrow{\mathcal{C}}$ under α ; see [H1, Def. 11.8.3]) containing all of the objects except the identity map of α .

- (5) The *matching object* of \mathbf{X} at α is

$$M_\alpha \mathbf{X} = \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} \mathbf{X}$$

and the *matching map* of \mathbf{X} at α is the natural map

$$\mathbf{X}_\alpha \longrightarrow M_\alpha \mathbf{X} .$$

We will use $M_\alpha^{\mathcal{C}} \mathbf{X}$ to denote the matching object if the indexing category is not obvious.

- (6) The *relative matching map* of $f: \mathbf{X} \rightarrow \mathbf{Y}$ at α is the map

$$\mathbf{X}_\alpha \longrightarrow \mathbf{Y}_\alpha \times_{M_\alpha \mathbf{Y}} M_\alpha \mathbf{X} .$$

Theorem 2.5 ([H1, Def. 15.3.3 and Thm. 15.3.4]). *Let \mathcal{C} be a Reedy category and let \mathcal{M} be a model category. There is a model category structure on the category $\mathcal{M}^{\mathcal{C}}$ of \mathcal{C} -diagrams in \mathcal{M} , called the Reedy model category structure, in which a map $f: \mathbf{X} \rightarrow \mathbf{Y}$ of \mathcal{C} -diagrams in \mathcal{M} is*

- a weak equivalence if for every object α of \mathcal{C} the map $f_\alpha: \mathbf{X}_\alpha \rightarrow \mathbf{Y}_\alpha$ is a weak equivalence in \mathcal{M} ,
- a cofibration if for every object α of \mathcal{C} the relative latching map $\mathbf{X}_\alpha \amalg_{L_\alpha \mathbf{X}} L_\alpha \mathbf{Y} \rightarrow \mathbf{Y}_\alpha$ (see Definition 2.4) is a cofibration in \mathcal{M} , and
- a fibration if for every object α of \mathcal{C} the relative matching map $\mathbf{X}_\alpha \rightarrow \mathbf{Y}_\alpha \times_{M_\alpha \mathbf{Y}} M_\alpha \mathbf{X}$ (see Definition 2.4) is a fibration in \mathcal{M} .

We also record the following standard result; we will have use for it in the proof of Proposition 2.17.

Proposition 2.6. *If \mathcal{M} is a category and $f: X \rightarrow Y$ is a map in \mathcal{M} , then f is an isomorphism if and only if it induces an isomorphism of the sets of maps $f_*: \mathcal{M}(W, X) \rightarrow \mathcal{M}(W, Y)$ for every object W of \mathcal{M} .*

Proof. If $g: Y \rightarrow X$ is an inverse for f , then $g_*: \mathcal{M}(W, Y) \rightarrow \mathcal{M}(W, X)$ is an inverse for f_* .

Conversely, if $f_*: \mathcal{M}(W, X) \rightarrow \mathcal{M}(W, Y)$ is an isomorphism for every object W of \mathcal{M} , then $f_*: \mathcal{M}(Y, X) \rightarrow \mathcal{M}(Y, Y)$ is an epimorphism, and so there is a map $g: Y \rightarrow X$ such that $fg = 1_Y$. We then have two maps $gf, 1_X: X \rightarrow X$, and

$$f_*(gf) = fgf = 1_Y f = f = f_*(1_X) .$$

Since $f_*: \mathcal{M}(X, X) \rightarrow \mathcal{M}(X, Y)$ is a monomorphism, this implies that $gf = 1_X$. \square

2.2. Filtrations of Reedy categories. The notion of a filtration of a Reedy category will be used in the proof of Theorem 4.3.

Definition 2.7. If \mathcal{C} is a Reedy category (with a chosen degree function) and n is a nonnegative integer, the n 'th *filtration* $F^n \mathcal{C}$ of \mathcal{C} is the full subcategory of \mathcal{C} with objects the objects of \mathcal{C} of degree at most n .

The following is a direct consequence of the definitions.

Proposition 2.8. *If \mathcal{C} is a Reedy category then each of its filtrations $F^n \mathcal{C}$ is a Reedy category with $\overrightarrow{F^n \mathcal{C}} = \overrightarrow{\mathcal{C}} \cap F^n \mathcal{C}$ and $\overleftarrow{F^n \mathcal{C}} = \overleftarrow{\mathcal{C}} \cap F^n \mathcal{C}$, and \mathcal{C} equals the union of the increasing sequence of subcategories $F^0 \mathcal{C} \subset F^1 \mathcal{C} \subset F^2 \mathcal{C} \subset \dots$.*

The following will be used in the proof of Theorem 4.3 (which is one direction of Theorem 1.1).

Proposition 2.9 ([H1, Thm. 15.2.1 and Cor. 15.2.9]). *For $n > 0$, extending a diagram \mathbf{X} on $F^{n-1}\mathcal{D}$ to one on $F^n\mathcal{D}$ consists of choosing, for every object γ of degree n , an object \mathbf{X}_γ and a factorization $L_\gamma\mathbf{X} \rightarrow \mathbf{X}_\gamma \rightarrow M_\gamma\mathbf{X}$ of the natural map $L_\gamma\mathbf{X} \rightarrow M_\gamma\mathbf{X}$ from the latching object of \mathbf{X} at γ to the matching object of \mathbf{X} at γ .*

2.3. Reedy functors.

Definition 2.10. If \mathcal{C} and \mathcal{D} are Reedy categories, then a *Reedy functor* $G: \mathcal{C} \rightarrow \mathcal{D}$ is a functor that takes $\overrightarrow{\mathcal{C}}$ into $\overrightarrow{\mathcal{D}}$ and takes $\overleftarrow{\mathcal{C}}$ into $\overleftarrow{\mathcal{D}}$. If \mathcal{D} is a Reedy category, then a *Reedy subcategory* of \mathcal{D} is a subcategory \mathcal{C} of \mathcal{D} that is a Reedy category for which the inclusion functor $\mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor.

Note that a Reedy functor is *not* required to respect the filtrations on the Reedy categories \mathcal{C} and \mathcal{D} (see Definition 2.7). Thus, a Reedy functor might take non-identity maps to identity maps. That case will be analyzed in Proposition 2.17; the other case will use the following proposition, whose proof follows immediately from the definitions.

Proposition 2.11. *Let \mathcal{C} and \mathcal{D} be Reedy categories, let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a Reedy functor, and let α be an object in \mathcal{C} .*

- (1) *If G takes every non-identity map $\alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$ to a non-identity map in $\overleftarrow{\mathcal{D}}$, then there is an induced functor of matching categories $G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \rightarrow \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ (see Definition 2.4) that takes the object $\sigma: \alpha \rightarrow \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ to the object $G\sigma: G\alpha \rightarrow G\gamma$ of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$.*
- (2) *If G takes every non-identity map $\gamma \rightarrow \alpha$ in $\overrightarrow{\mathcal{C}}$ to a non-identity map in $\overrightarrow{\mathcal{D}}$, then there is an induced functor of latching categories $G_*: \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \rightarrow \partial(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$ (see Definition 2.4) that takes the object $\sigma: \gamma \rightarrow \alpha$ of $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ to the object $G\sigma: G\gamma \rightarrow G\alpha$ of $\partial(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$.*

Definition 2.12. If $G: \mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor between Reedy categories and \mathcal{M} is a model category, then G induces a functor of diagram categories $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ under which

- a functor $\mathbf{X}: \mathcal{D} \rightarrow \mathcal{M}$ goes to the functor $G^*\mathbf{X}: \mathcal{C} \rightarrow \mathcal{M}$ that is the composition $\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{\mathbf{X}} \mathcal{M}$ (so that for an object α of \mathcal{C} we have $(G^*\mathbf{X})_\alpha = \mathbf{X}_{G\alpha}$) and
- a natural transformation of \mathcal{D} -diagrams $f: \mathbf{X} \rightarrow \mathbf{Y}$ goes to the natural transformation of \mathcal{C} -diagrams G^*f that on a map $\sigma: \alpha \rightarrow \beta$ of \mathcal{C} is the map $(G\sigma)_*: \mathbf{X}_{G\alpha} \rightarrow \mathbf{Y}_{G\alpha}$ in \mathcal{M} .

The main results of this paper (Theorem 1.1 and Theorem 1.2) determine when the functor $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ is either a left Quillen functor or a right Quillen functor. The answer to those questions will depend on the notions of the *category of inverse \mathcal{C} -factorizations* of a map in $\overleftarrow{\mathcal{D}}$ and the *category of direct \mathcal{C} -factorizations* of a map in $\overrightarrow{\mathcal{D}}$.

Definition 2.13. Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a Reedy functor between Reedy categories, let α be an object of \mathcal{C} , and let β be an object of \mathcal{D} .

- (1) If $\sigma: G\alpha \rightarrow \beta$ is a map in $\overleftarrow{\mathcal{D}}$, then the *category of inverse \mathcal{C} -factorizations* of (α, σ) is the category in which

- an object is a pair

$$((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$$

consisting of a non-identity map $\nu: \alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$ and a map $\mu: G\gamma \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ such that the composition $G\alpha \xrightarrow{G\nu} G\gamma \xrightarrow{\mu} \beta$ equals σ , and

- a map from $((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$ to $((\nu': \alpha \rightarrow \gamma'), (\mu': G\gamma' \rightarrow \beta))$ is a map $\tau: \gamma \rightarrow \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that the triangles

$$\begin{array}{ccc} & \alpha & \\ \nu \swarrow & & \searrow \nu' \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array} \quad \text{and} \quad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ \mu \searrow & & \swarrow \mu' \\ & \beta & \end{array}$$

commute.

We will often refer just to the map σ when the object α is obvious. In particular, when $G: \mathcal{C} \rightarrow \mathcal{D}$ is the inclusion of a subcategory the object α is determined by the morphism σ , and we will often refer to the *category of inverse \mathcal{C} -factorizations of σ* .

- (2) If $\sigma: \beta \rightarrow G\alpha$ is a map in $\overrightarrow{\mathcal{D}}$, then the *category of direct \mathcal{C} -factorizations of (α, σ)* is the category in which

- an object is a pair

$$((\nu: \gamma \rightarrow \alpha), (\mu: \beta \rightarrow G\gamma))$$

consisting of a non-identity map $\nu: \gamma \rightarrow \alpha$ in $\overrightarrow{\mathcal{C}}$ and a map $\mu: \beta \rightarrow G\gamma$ in $\overrightarrow{\mathcal{D}}$ such that the composition $\beta \xrightarrow{\mu} G\gamma \xrightarrow{G\nu} G\alpha$ equals σ , and

- a map from $((\nu: \gamma \rightarrow \alpha), (\mu: \beta \rightarrow G\gamma))$ to $((\nu': \gamma' \rightarrow \alpha), (\mu': \beta \rightarrow G\gamma'))$ is a map $\tau: \gamma \rightarrow \gamma'$ in $\overrightarrow{\mathcal{C}}$ such that the triangles

$$\begin{array}{ccc} & \alpha & \\ \nu \nearrow & & \nwarrow \gamma' \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array} \quad \text{and} \quad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ \mu \nwarrow & & \nearrow \mu' \\ & \beta & \end{array}$$

commute.

We will often refer just to the map σ when the object α is obvious. In particular, when $G: \mathcal{C} \rightarrow \mathcal{D}$ is the inclusion of a subcategory the object α is determined by the morphism σ , and we will often refer to the *category of direct \mathcal{C} -factorizations of σ* .

Proposition 2.14. *Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a Reedy functor between Reedy categories, let α be an object of \mathcal{C} , and let β be an object of \mathcal{D} .*

- (1) *If $\sigma: G\alpha \rightarrow \beta$ is a map in $\overleftarrow{\mathcal{D}}$, then we have an induced functor*

$$G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \longrightarrow (G\alpha \downarrow \overleftarrow{\mathcal{D}})$$

from the matching category of \mathcal{C} at α to the category of objects of $\overleftarrow{\mathcal{D}}$ under $G\alpha$ that takes the object $\alpha \rightarrow \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ to the object $G\alpha \rightarrow G\gamma$ of $(G\alpha \downarrow \overleftarrow{\mathcal{D}})$, and the category of inverse \mathcal{C} -factorizations of (α, σ) (see Definition 2.13) is the category $(G_ \downarrow \sigma)$ of objects of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ over σ .*

- (2) *If $\sigma: \beta \rightarrow G\alpha$ is a map in $\overrightarrow{\mathcal{D}}$, then we have an induced functor*

$$G_*: \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \longrightarrow (\overrightarrow{\mathcal{D}} \downarrow G\alpha)$$

from the latching category of \mathcal{C} at α to the category of objects of $\vec{\mathcal{D}}$ over $G\alpha$ that takes the object $\gamma \rightarrow \alpha$ of $\partial(\vec{\mathcal{C}} \downarrow \alpha)$ to the object $G\gamma \rightarrow G\alpha$ of $(\vec{\mathcal{D}} \downarrow G\alpha)$, and the category of direct \mathcal{C} -factorizations of (α, σ) is the category $(\sigma \downarrow G_*)$ of objects of $\partial(\vec{\mathcal{C}} \downarrow \alpha)$ under σ .

Proof. We will prove part 1; the proof of part 2 is similar. An object of $(G_* \downarrow \sigma)$ is a pair $((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$ where $\nu: \alpha \rightarrow \gamma$ is an object of $\partial(\alpha \downarrow \vec{\mathcal{C}})$ and $\mu: G\gamma \rightarrow \beta$ is a map in $\vec{\mathcal{D}}$ that makes the triangle

$$\begin{array}{ccc} & G\alpha & \\ G\nu \swarrow & & \searrow \sigma \\ G\gamma & \xrightarrow{\mu} & \beta \end{array}$$

commute. A map from $((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$ to $((\nu': \alpha \rightarrow \gamma'), (\mu': G\gamma' \rightarrow \beta'))$ is a map $\tau: \gamma \rightarrow \gamma'$ in $\vec{\mathcal{C}}$ that makes the triangles

$$\begin{array}{ccc} & \alpha & \\ \nu \swarrow & & \searrow \nu' \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array} \quad \text{and} \quad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ \mu \searrow & & \swarrow \mu' \\ & \beta & \end{array}$$

commute. This is exactly the definition of the category of inverse \mathcal{C} -factorizations of (α, σ) . \square

Proposition 2.15. *Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a Reedy functor between Reedy categories, let α be an object of \mathcal{C} , and let β be an object of \mathcal{D} .*

- (1) *If $\sigma: G\alpha \rightarrow \beta$ is a map in $\vec{\mathcal{D}}$ and G takes every non-identity map $\alpha \rightarrow \gamma$ in $\vec{\mathcal{C}}$ to a non-identity map in $\vec{\mathcal{D}}$, then we have an induced functor*

$$G_*: \partial(\alpha \downarrow \vec{\mathcal{C}}) \longrightarrow \partial(G\alpha \downarrow \vec{\mathcal{D}})$$

of matching categories (see Proposition 2.11), and the category of inverse \mathcal{C} -factorizations of (α, σ) (see Definition 2.13) is the category $(G_ \downarrow \sigma)$ of objects of $\partial(\alpha \downarrow \vec{\mathcal{C}})$ over σ .*

- (2) *If $\sigma: \beta \rightarrow G\alpha$ is a map in $\vec{\mathcal{D}}$ and G takes every non-identity map $\gamma \rightarrow \alpha$ in $\vec{\mathcal{C}}$ to a non-identity map in $\vec{\mathcal{D}}$, then we have an induced functor*

$$G_*: \partial(\vec{\mathcal{C}} \downarrow \alpha) \longrightarrow \partial(\vec{\mathcal{D}} \downarrow G\alpha)$$

of latching categories (see Proposition 2.11), and the category of direct \mathcal{C} -factorizations of (α, σ) is the category $(\sigma \downarrow G_)$ of objects of $\partial(\vec{\mathcal{C}} \downarrow \alpha)$ under σ .*

Proof. This is identical to the proof of Proposition 2.14, except that the requirement that certain non-identity maps go to non-identity maps ensures (in part 1) that the functor $G_*: \partial(\alpha \downarrow \vec{\mathcal{C}}) \rightarrow (G\alpha \downarrow \vec{\mathcal{D}})$ factors through the subcategory $\partial(G\alpha \downarrow \vec{\mathcal{D}})$ of $(G\alpha \downarrow \vec{\mathcal{D}})$ and (in part 2) that the functor $G_*: \partial(\vec{\mathcal{C}} \downarrow \alpha) \rightarrow (\vec{\mathcal{D}} \downarrow G\alpha)$ factors through the subcategory $\partial(\vec{\mathcal{D}} \downarrow G\alpha)$ of $(\vec{\mathcal{D}} \downarrow G\alpha)$. \square

Definition 2.16. Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a Reedy functor between Reedy categories.

- (1) The Reedy functor G is a *fibering Reedy functor* if for every object α in \mathcal{C} , every object β in \mathcal{D} , and every map $\sigma: G\alpha \rightarrow \beta$ in $\vec{\mathcal{D}}$, the nerve of the

category of inverse \mathcal{C} -factorizations of (α, σ) (see Definition 2.13) is either empty or connected.

If \mathcal{C} is a Reedy subcategory of \mathcal{D} and if the inclusion is a fibering Reedy functor, then we will call \mathcal{C} a *fibering Reedy subcategory* of \mathcal{D} .

- (2) The Reedy functor G is a *cofibering Reedy functor* if for every object α in \mathcal{C} , every object β in \mathcal{D} , and every map $\sigma: \beta \rightarrow G\alpha$ in $\overrightarrow{\mathcal{D}}$, the nerve of the category of direct \mathcal{C} -factorizations of (α, σ) (see Definition 2.13) is either empty or connected.

If \mathcal{C} is a Reedy subcategory of \mathcal{D} and if the inclusion is a cofibering Reedy functor, then we will call \mathcal{C} a *cofibering Reedy subcategory* of \mathcal{D} .

Examples of fibering Reedy functors and of cofibering Reedy functors (and of Reedy functors that are not fibering) are given in Section 3.

The following proposition will be used in the proof of Theorem 4.2. Its own proof requires several preliminary definitions and results, which we have collected in the next section.

Proposition 2.17. *Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a fibering Reedy functor and let \mathbf{X} be a \mathcal{D} -diagram in a model category \mathcal{M} . If α is an object of \mathcal{C} for which there is an object $\alpha \rightarrow \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ (i.e., a non-identity map $\alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$) that G takes to an identity map in $\overleftarrow{\mathcal{D}}$, then the matching map $(G^*\mathbf{X})_\alpha \rightarrow M_\alpha^{\mathcal{C}}(G^*\mathbf{X})$ of $G^*\mathbf{X}$ (see Definition 2.12) at α is an isomorphism.*

2.3.1. Proof of Proposition 2.17. We will use the term *G -kernel at α* to refer to the full subcategory of the matching category $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ with objects the non-identity maps $\alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$ that G takes to the identity map of $G\alpha$. If $\alpha \rightarrow \gamma$ is an object of the G -kernel at α , then the map $(G^*\mathbf{X})_\alpha \rightarrow (G^*\mathbf{X})_\gamma$ is the identity map.

Lemma 2.18. *The nerve of the G -kernel at α is connected.*

Proof. There is an isomorphism from the G -kernel at α to the category of inverse \mathcal{C} -factorizations of $(\alpha, 1_{G\alpha})$ that takes the object $\alpha \rightarrow \gamma$ to the object $((\alpha \rightarrow \gamma), (1_{G\alpha}))$. Since G is a fibering Reedy functor, the nerve of the category of inverse \mathcal{C} -factorizations of $(\alpha, 1_{G\alpha})$ is connected. \square

The matching object $M_\alpha^{\mathcal{C}}(G^*\mathbf{X})$ is the limit of a $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ -diagram (which we will also denote by $G^*\mathbf{X}$); we will refer to that diagram as the *matching diagram*. The restriction of the matching diagram to the G -kernel at α is a diagram in which every object goes to $\mathbf{X}_{G\alpha} = (G^*\mathbf{X})_\alpha$ and every map goes to the identity map of $\mathbf{X}_{G\alpha}$ (because if there is a commutative triangle

$$\begin{array}{ccc} & \alpha & \\ f \swarrow & & \searrow f' \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array}$$

in $\overleftarrow{\mathcal{C}}$ in which $Gf = Gf' = 1_{G\alpha}$, then $G\tau$ is a map in $\overleftarrow{\mathcal{D}}$ that does not lower degree, and so $G\tau$ must be an identity map). Together with Lemma 2.18, this implies the following.

Lemma 2.19. *The restriction of the matching diagram to the G -kernel at α is a diagram in which every object goes to $\mathbf{X}_{G\alpha}$, every map goes to the identity map of $\mathbf{X}_{G\alpha}$, and every two objects are connected by a zig-zag of maps.*

We define an equivalence relation on the set of objects of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$, called *G-equivalence at α* , as the equivalence relation generated by the relation under which $f: \alpha \rightarrow \gamma$ is equivalent to $f': \alpha \rightarrow \gamma'$ if there is a commutative triangle

$$\begin{array}{ccc} & \alpha & \\ f \swarrow & & \searrow f' \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array}$$

with $G\tau$ an identity map. Thus, if f and f' are G -equivalent at α , then $Gf = Gf'$, and there is a zig-zag of identity maps connecting \mathbf{X}_f and $\mathbf{X}_{f'}$ in the matching diagram.

Definition 2.20. We define the set of *controlled objects* $\{\alpha \rightarrow \gamma\}$ of the matching category $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ by a decreasing induction on $\text{degree}(G\gamma)$:

- (1) If $\alpha \rightarrow \gamma$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ such that $\text{degree}(G\gamma) = \text{degree}(G\alpha)$, then $\alpha \rightarrow \gamma$ is controlled. (That is, all objects of the G -kernel at α are controlled.)
- (2) If $0 \leq n < \text{degree}(G\alpha)$ and we have defined the controlled objects $\alpha \rightarrow \delta$ for $n < \text{degree}(\delta) \leq \text{degree}(G\alpha)$, then we define an object $\alpha \rightarrow \gamma$ with $\text{degree}(G\gamma) = n$ to be controlled if it is G -equivalent at α to an object $\alpha \rightarrow \gamma'$ that has a factorization $\alpha \rightarrow \delta \rightarrow \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that $\alpha \rightarrow \delta$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ that is controlled.

Example 2.21. Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be the fibering Reedy functor between Reedy categories as in the following diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ \begin{array}{c} \alpha \\ \sigma \swarrow \quad \searrow \beta \\ \gamma \\ \tau \downarrow \\ \delta \quad \mu \searrow \epsilon \end{array} & & \begin{array}{c} a \\ \downarrow f \\ b \end{array} \end{array}$$

where

- \mathcal{C} has five objects, $\alpha, \beta, \gamma, \delta$, and ϵ of degrees 4, 3, 2, 1, and 0, respectively, and the diagram commutes;
- \mathcal{D} has two objects, a and b of degrees 1 and 0, respectively;
- $G\alpha = G\beta = G\gamma = a$ and G takes the maps between them to 1_a ;
- $G\delta = G\epsilon = b$ and $G\mu = 1_b$; and
- $G\sigma = G\tau = f$.

Every object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ is controlled:

- The objects $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$ are controlled because of the first part of Definition 2.20.
- The object σ is controlled because it is G -equivalent at α to $\alpha \rightarrow \epsilon$ and the latter map factors as $\alpha \rightarrow \gamma \rightarrow \epsilon$ where the object $\alpha \rightarrow \gamma$ is controlled.
- The object $\alpha \rightarrow \epsilon$ is controlled for the same reason; it is G -equivalent at α to itself and it factors as $\alpha \rightarrow \gamma \rightarrow \epsilon$ with the object $\alpha \rightarrow \gamma$ controlled.

If \mathbf{X} is a \mathcal{D} -diagram in a model category \mathcal{M} , then the induced \mathcal{C} -diagram $G^*\mathbf{X}$ has

$$(G^*\mathbf{X})_\alpha = (G^*\mathbf{X})_\beta = (G^*\mathbf{X})_\gamma = \mathbf{X}_a \quad \text{and} \quad (G^*\mathbf{X})_\delta = (G^*\mathbf{X})_\epsilon = \mathbf{X}_b ,$$

and the matching object of $(G^*\mathbf{X})$ at α is the limit of the diagram

$$\begin{array}{ccc} & & \mathbf{X}_a \\ & \swarrow & \downarrow 1_{\mathbf{X}_a} \\ & \mathbf{X}_a & \\ \mathbf{X}_b & \searrow & \downarrow \mathbf{X}_f \\ & \mathbf{X}_b & \end{array} ;$$

that limit is isomorphic to \mathbf{X}_a .

The set of controlled objects has the following property.

Lemma 2.22. *If W is an object of \mathcal{M} and $h, k: W \rightarrow M_\alpha^{\mathcal{C}}(G^*\mathbf{X})$ are two maps to the matching object of $G^*\mathbf{X}$ at α whose projections onto at least one object of the G -kernel at α agree, then their projections onto every controlled object agree.*

Proof. This follows by a decreasing induction as in Definition 2.20, using Lemma 2.19 and Definition 2.20. \square

That every object in the example above was controlled was not an accident, as shown by the following result.

Proposition 2.23. *Every object $f: \alpha \rightarrow \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ is controlled.*

Proof. We will show this by a decreasing induction on the degree of $G\gamma$ in \mathcal{D} , beginning with $\text{degree}(G\alpha)$. The induction is begun because the objects $f: \alpha \rightarrow \gamma$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ with $\text{degree}(G\gamma) = \text{degree}(G\alpha)$ are exactly the objects of the G -kernel at α , since a map in $\overleftarrow{\mathcal{D}}$ that does not lower degree must be an identity map.

Suppose now that $0 \leq n < \text{degree}(G\alpha)$, that every object $\alpha \rightarrow \delta$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ with $\text{degree}(G\delta) > n$ is controlled, and that $f: \alpha \rightarrow \gamma$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ with $\text{degree}(G\gamma) = n$. Consider the category of inverse \mathcal{C} -factorizations of $(\alpha, Gf: G\alpha \rightarrow G\gamma)$. That category contains the object $((f: \alpha \rightarrow \gamma), (1_{G\gamma}))$ and, if $g: \alpha \rightarrow \delta$ is an object of the G -kernel at α , then it also contains the object $((g: \alpha \rightarrow \delta), (Gf: G\alpha \rightarrow G\gamma))$. Since G is a fibering Reedy functor, there must be a zig-zag of maps in the category of inverse \mathcal{C} -factorizations of (α, Gf) connecting those two objects.

If $f': \alpha \rightarrow \gamma'$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ that is G -equivalent at α to $f: \alpha \rightarrow \gamma$, then $Gf' = Gf$, and if there is a map in the category of inverse \mathcal{C} -factorizations of (α, Gf) from $((f': \alpha \rightarrow \gamma'), (1_{G\gamma}))$ to another object, then that other object must be of the form $((f'': \alpha \rightarrow \gamma''), (1_{G\gamma}))$ where f'' is also G -equivalent at α to f . This is because if $\tau: \gamma' \rightarrow \gamma''$ is a map in $\overleftarrow{\mathcal{C}}$ such that $G\tau$ is *not* an identity map, then $\text{degree}(G\gamma'') < \text{degree}(G\gamma') = \text{degree}(G\gamma)$ and so there is no object of the category of inverse \mathcal{C} -factorizations of (α, Gf) of the form $((\tau f': \alpha \rightarrow \gamma''), (G\gamma'' \rightarrow G\gamma))$. Thus, if $((f': \alpha \rightarrow \gamma'), (1_{G\gamma}))$ is an object of the category of inverse \mathcal{C} -factorizations of (α, Gf) such that f' is G -equivalent at α to f , then that object is not the domain of any map to an object $((h: \alpha \rightarrow \epsilon), (G\epsilon \rightarrow G\gamma))$ with $\text{degree}(G\epsilon) \neq \text{degree}(G\gamma)$.

If $((h: \alpha \rightarrow \epsilon), (G\epsilon \rightarrow G\gamma))$ is an object of the category of inverse \mathcal{C} -factorizations of (α, Gf) and $\tau: \epsilon \rightarrow \gamma'$ is a map to $((g': \alpha \rightarrow \gamma'), (1_{G\gamma'}))$, then either $\text{degree}(G\epsilon) > \text{degree}(G\gamma)$ or $G\tau$ must be an identity map (because if $\text{degree}(G\epsilon) = \text{degree}(G\gamma)$ then there is a commutative triangle

$$\begin{array}{ccc} G\epsilon & \xrightarrow{G\tau} & G\gamma' \\ & \searrow & \swarrow 1_{G\gamma'} \\ & G\gamma' & \end{array}$$

in which the map $G\epsilon \rightarrow G\gamma'$ is a map of $\overleftarrow{\mathcal{D}}$ that does not lower degree and is thus an identity map.) Thus, the only way an object G -equivalent to f can connect to an object connected to $((g: \alpha \rightarrow \delta), (Gf: G\alpha \rightarrow G\gamma))$ is by way of a map $\tau: \epsilon \rightarrow \gamma'$ from an object $((h: \alpha \rightarrow \epsilon), (G\epsilon \rightarrow G\gamma))$ with $\text{degree}(G\epsilon) > \text{degree}(G\gamma)$, which (by the induction hypothesis) implies that $h: \alpha \rightarrow \epsilon$ is controlled. In this case, the composition $\alpha \xrightarrow{h} \epsilon \xrightarrow{\tau} \gamma'$ equals $f': \alpha \rightarrow \gamma'$, and so $f: \alpha \rightarrow \gamma$ is controlled. This completes the induction. \square

Proof of Proposition 2.17. Proposition 2.6 implies that it is sufficient to show that for every object W of \mathcal{M} the matching map $(G^*\mathbf{X})_\alpha \rightarrow M_\alpha^{\mathcal{C}}(G^*\mathbf{X})$ induces an isomorphism of the sets of maps

$$\mathcal{M}(W, (G^*\mathbf{X})_\alpha) \approx \mathcal{M}(W, M_\alpha^{\mathcal{C}}(G^*\mathbf{X})) .$$

Let W be an object of \mathcal{M} and let $h: W \rightarrow M_\alpha^{\mathcal{C}}(G^*\mathbf{X})$ be a map. If $\alpha \rightarrow \gamma$ is an object of the G -kernel at α of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$, then $(G^*\mathbf{X})_{(\alpha \rightarrow \gamma)} = (G^*\mathbf{X})_\gamma = (G^*\mathbf{X})_\alpha$, and so the projection of h onto $(G^*\mathbf{X})_{(\alpha \rightarrow \gamma)}$ defines a map $\hat{h}: W \rightarrow (G^*\mathbf{X})_\alpha$. Lemma 2.19 implies that the map \hat{h} is independent of the choice of object of the identity subcategory.

The composition

$$W \xrightarrow{\hat{h}} (G^*\mathbf{X})_\alpha \longrightarrow M_\alpha^{\mathcal{C}}(G^*\mathbf{X})$$

has the same projection onto $(G^*\mathbf{X})_{(\alpha \rightarrow \gamma)}$ as the map $h: W \rightarrow M_\alpha^{\mathcal{C}}(G^*\mathbf{X})$; since every object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ is controlled (see Proposition 2.23), these two maps agree on every projection of $M_\alpha^{\mathcal{C}}(G^*\mathbf{X})$ (see Lemma 2.22), and so they are equal. Since the composition of the matching map with the projection $M_\alpha^{\mathcal{C}}(G^*\mathbf{X}) \rightarrow (G^*\mathbf{X})_{(\alpha \rightarrow \gamma)}$ is the identity map, \hat{h} is the only possible lift to $(G^*\mathbf{X})_\alpha$ of h , and so the matching map induces an isomorphism of the set of maps $\mathcal{M}(W, (G^*\mathbf{X})_\alpha) \approx \mathcal{M}(W, M_\alpha^{\mathcal{C}}(G^*\mathbf{X}))$. Thus, the matching map is an isomorphism. \square

2.4. Opposites. The results in this section will be used in the proof of Theorem 1.2, which can be found in Section 4.3.

Proposition 2.24. *If \mathcal{C} is a Reedy category, then the opposite category \mathcal{C}^{op} is a Reedy category in which $\overrightarrow{\mathcal{C}^{\text{op}}} = (\overleftarrow{\mathcal{C}})^{\text{op}}$ and $\overleftarrow{\mathcal{C}^{\text{op}}} = (\overrightarrow{\mathcal{C}})^{\text{op}}$.*

Proof. A degree function for \mathcal{C} will serve as a degree function for \mathcal{C}^{op} , and factorizations $\sigma = \tau\mu$ in \mathcal{C} with $\mu \in \overleftarrow{\mathcal{C}}$ and $\tau \in \overrightarrow{\mathcal{C}}$ correspond to factorizations $\sigma^{\text{op}} = \mu^{\text{op}}\tau^{\text{op}}$ in \mathcal{C}^{op} with $\mu^{\text{op}} \in (\overleftarrow{\mathcal{C}})^{\text{op}} = \overrightarrow{\mathcal{C}^{\text{op}}}$ and $\tau^{\text{op}} \in (\overrightarrow{\mathcal{C}})^{\text{op}} = \overleftarrow{\mathcal{C}^{\text{op}}}$. \square

Proposition 2.25. *If \mathcal{C} and \mathcal{D} are Reedy categories, then a functor $G: \mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor if and only if its opposite $G^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ is a Reedy functor.*

Proof. This follows from Proposition 2.24. \square

Lemma 2.26. *Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a Reedy functor between Reedy categories, let α be an object of \mathcal{C} , and let β be an object of \mathcal{D} .*

- (1) *If $\sigma: G\alpha \rightarrow \beta$ is a map in $\overleftarrow{\mathcal{D}}$, then the opposite of the category of inverse \mathcal{C} -factorizations of (α, σ) is the category of direct \mathcal{C}^{op} -factorizations of $(\alpha, \sigma^{\text{op}}: \beta \rightarrow G\alpha)$ in $\overrightarrow{\mathcal{D}^{\text{op}}}$.*
- (2) *If $\sigma: \beta \rightarrow G\alpha$ is a map in $\overrightarrow{\mathcal{D}}$, then the opposite of the category of direct \mathcal{C} -factorizations of (α, σ) is the category of inverse \mathcal{C}^{op} -factorizations of $(\alpha, \sigma^{\text{op}}: G\alpha \rightarrow \beta)$ in $\overleftarrow{\mathcal{D}^{\text{op}}}$.*

Proof. We will prove part (1); part (2) will then follow from applying part (1) to $\sigma^{\text{op}}: G\alpha \rightarrow \beta$ in \mathcal{C}^{op} and remembering that $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$ and $(\mathcal{D}^{\text{op}})^{\text{op}} = \mathcal{D}$.

Let $\sigma: G\alpha \rightarrow \beta$ be a map in $\overleftarrow{\mathcal{D}}$. Recall from Definition 2.13 that

- an object of the category of inverse \mathcal{C} -factorizations of $(\alpha, \sigma: G\alpha \rightarrow \beta)$ is a pair

$$((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$$

consisting of a non-identity map $\nu: \alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$ and a map $\mu: G\gamma \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ such that the composition $G\alpha \xrightarrow{G\nu} G\gamma \xrightarrow{\mu} \beta$ equals σ , and

- a map from $((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$ to $((\nu': \alpha \rightarrow \gamma'), (\mu': G\gamma' \rightarrow \beta))$ is a map $\tau: \gamma \rightarrow \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that the triangles

$$\begin{array}{ccc} & \alpha & \\ \nu \swarrow & & \searrow \nu' \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array} \quad \text{and} \quad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ \mu \searrow & & \swarrow \mu' \\ & \beta & \end{array}$$

commute.

The opposite of this category has the same objects, but

- a non-identity map $\nu: \alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$ is equivalently a non-identity map $\nu^{\text{op}}: \gamma \rightarrow \alpha$ in $(\overleftarrow{\mathcal{C}})^{\text{op}} = \overrightarrow{\mathcal{C}^{\text{op}}}$, and
- a factorization $G\alpha \xrightarrow{G\nu} G\gamma \xrightarrow{\mu} \beta$ of σ such that $\mu \in \overleftarrow{\mathcal{D}}$ is equivalently a factorization $\beta \xrightarrow{\mu^{\text{op}}} G\gamma \xrightarrow{G\nu^{\text{op}}} G\alpha$ of $\sigma^{\text{op}}: \beta \rightarrow G\alpha$ in $(\overleftarrow{\mathcal{D}})^{\text{op}} = \overrightarrow{\mathcal{D}^{\text{op}}}$

Thus, the opposite category can be described as the category in which

- An object is a pair

$$((\nu^{\text{op}}: \gamma \rightarrow \alpha), (\mu^{\text{op}}: \beta \rightarrow G\gamma))$$

consisting of a non-identity map $\nu^{\text{op}}: \gamma \rightarrow \alpha$ in $(\overleftarrow{\mathcal{C}})^{\text{op}} = \overrightarrow{\mathcal{C}^{\text{op}}}$ and a map $\mu^{\text{op}}: \beta \rightarrow G\gamma$ in $(\overleftarrow{\mathcal{D}})^{\text{op}} = \overrightarrow{\mathcal{D}^{\text{op}}}$ such that the composition $\beta \xrightarrow{\mu^{\text{op}}} G\gamma \xrightarrow{G\nu^{\text{op}}} G\alpha$ equals σ^{op} , and

- a map from $((\nu^{\text{op}}: \gamma \rightarrow \alpha), (\mu^{\text{op}}: \beta \rightarrow G\gamma))$ to $((\nu')^{\text{op}}: \gamma' \rightarrow \alpha), ((\mu')^{\text{op}}: \beta \rightarrow G\gamma')$ is a map $\tau^{\text{op}}: \gamma' \rightarrow \gamma$ in $(\overleftarrow{\mathcal{C}})^{\text{op}} = \mathcal{C}^{\text{op}}$ such that the triangles

$$\begin{array}{ccc} & \alpha & \\ \nu^{\text{op}} \nearrow & & \nwarrow (\nu')^{\text{op}} \\ \gamma & \xleftarrow{\tau^{\text{op}}} & \gamma' \end{array} \quad \text{and} \quad \begin{array}{ccc} G\gamma & \xleftarrow{G\tau^{\text{op}}} & G\gamma' \\ \mu^{\text{op}} \nearrow & & \nwarrow (\mu')^{\text{op}} \\ \beta & & \end{array}$$

commute.

This is exactly the category of direct \mathcal{C}^{op} -factorizations of $(\alpha, \sigma^{\text{op}}: \beta \rightarrow G\alpha)$ in $\overrightarrow{\mathcal{D}^{\text{op}}}$. \square

Proposition 2.27. *If $G: \mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor between Reedy categories, then G is a fibering Reedy functor if and only if $G^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ is a cofibering Reedy functor.*

Proof. Since the nerve of a category is empty or connected if and only if the nerve of the opposite category is, respectively, empty or connected, this follows from Lemma 2.26. \square

Lemma 2.28. *Let \mathbf{X} be a \mathcal{C} -diagram in \mathcal{M} (which can also be viewed as a \mathcal{C}^{op} -diagram in \mathcal{M}^{op}), and let α be an object of \mathcal{C} .*

- (1) *The latching object $L_\alpha^{\mathcal{C}}$ of \mathbf{X} as a \mathcal{C} -diagram in \mathcal{M} at α is the matching object $M_\alpha^{\mathcal{C}^{\text{op}}}$ of \mathbf{X} as a \mathcal{C}^{op} -diagram in \mathcal{M}^{op} at α , and the opposite of the latching map $L_\alpha^{\mathcal{C}}\mathbf{X} \rightarrow \mathbf{X}$ of \mathbf{X} as a \mathcal{C} -diagram in \mathcal{M} at α is the matching map $\mathbf{X} \rightarrow L_\alpha^{\mathcal{C}}\mathbf{X} = M_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{X}$ of \mathbf{X} as a \mathcal{C}^{op} -diagram in \mathcal{M}^{op} at α .*
- (2) *The matching object $M_\alpha^{\mathcal{C}}$ of \mathbf{X} as a \mathcal{C} -diagram in \mathcal{M} at α is the latching object $L_\alpha^{\mathcal{C}^{\text{op}}}$ of \mathbf{X} as a \mathcal{C}^{op} -diagram in \mathcal{M}^{op} at α , and the opposite of the matching map $\mathbf{X} \rightarrow M_\alpha^{\mathcal{C}}\mathbf{X}$ of \mathbf{X} as a \mathcal{C} -diagram in \mathcal{M} at α is the latching map $L_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{X} = M_\alpha^{\mathcal{C}}\mathbf{X} \rightarrow \mathbf{X}$ of \mathbf{X} as a \mathcal{C}^{op} -diagram in \mathcal{M}^{op} at α .*

Proof. We will prove part 1; part 2 then follows by applying part 1 to the \mathcal{C}^{op} -diagram \mathbf{X} in \mathcal{M}^{op} and remembering that $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$ and $(\mathcal{M}^{\text{op}})^{\text{op}} = \mathcal{M}$.

The latching object $L_\alpha^{\mathcal{C}}\mathbf{X}$ of \mathbf{X} at α is the colimit of the diagram in \mathcal{M} with an object \mathbf{X}_β for every non-identity map $\sigma: \beta \rightarrow \alpha$ in $\overrightarrow{\mathcal{C}}$ and a map $\mu_*: \mathbf{X}_\beta \rightarrow \mathbf{X}_\gamma$ for every commutative triangle

$$\begin{array}{ccc} & \alpha & \\ \sigma \nearrow & & \nwarrow \tau \\ \beta & \xrightarrow{\mu} & \gamma \end{array}$$

in $\overrightarrow{\mathcal{C}}$ in which σ and τ are non-identity maps. Thus, $L_\alpha^{\mathcal{C}}\mathbf{X}$ can also be described as the limit of the diagram in \mathcal{M}^{op} with one object \mathbf{X}_β for every non-identity map $\sigma^{\text{op}}: \alpha \rightarrow \beta$ in $(\overrightarrow{\mathcal{C}})^{\text{op}} = \overleftarrow{\mathcal{C}^{\text{op}}}$ and a map $(\mu^{\text{op}})_*: \mathbf{X}_\gamma \rightarrow \mathbf{X}_\beta$ for every commutative triangle

$$\begin{array}{ccc} & \alpha & \\ \sigma^{\text{op}} \nwarrow & & \nearrow \tau^{\text{op}} \\ \beta & \xleftarrow{\mu^{\text{op}}} & \gamma \end{array}$$

in $(\overrightarrow{\mathcal{C}})^{\text{op}} = \overleftarrow{\mathcal{C}^{\text{op}}}$ in which σ^{op} and τ^{op} are non-identity maps. Thus, $L_\alpha^{\mathcal{C}}\mathbf{X} = M_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{X}$.

The latching map $L_\alpha^{\mathcal{C}}\mathbf{X} \rightarrow \mathbf{X}_\alpha$ is the unique map in \mathcal{M} such that for every non-identity map $\sigma: \beta \rightarrow \alpha$ in $\overrightarrow{\mathcal{C}}$ the triangle

$$\begin{array}{ccc} & \mathbf{X}_\alpha & \\ \sigma_* \nearrow & \uparrow & \\ \mathbf{X}_\beta & \rightarrow & L_\alpha^{\mathcal{C}}\mathbf{X} \end{array}$$

commutes, and so the opposite of the latching map is the unique map $\mathbf{X}_\alpha \rightarrow L_\alpha^{\mathcal{C}}\mathbf{X} = M_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{X}$ in \mathcal{M}^{op} such that for every non-identity map $\sigma^{\text{op}}: \alpha \rightarrow \beta$ in $(\overrightarrow{\mathcal{C}})^{\text{op}} = \overleftarrow{\mathcal{C}^{\text{op}}}$ the triangle

$$\begin{array}{ccc} & \mathbf{X}_\alpha & \\ (\sigma^{\text{op}})_* \nwarrow & \downarrow & \\ \mathbf{X}_\beta & \leftarrow & M_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{X} \end{array}$$

commutes, i.e., the opposite of the latching map of \mathbf{X} at α in \mathcal{C} is the matching map of \mathbf{X} at α in \mathcal{C}^{op} . \square

Lemma 2.29. *Let $f: \mathbf{X} \rightarrow \mathbf{Y}$ be a map of \mathcal{C} -diagrams in \mathcal{M} and let α be an object of \mathcal{C} .*

- (1) *The opposite of the relative latching map (see Definition 2.4) of f at α is the relative matching map of the map $f^{\text{op}}: \mathbf{Y} \rightarrow \mathbf{X}$ of \mathcal{C}^{op} -diagrams in \mathcal{M}^{op} at α .*
- (2) *The opposite of the relative matching map (see Definition 2.4) of f at α is the relative latching map of the map $f^{\text{op}}: \mathbf{Y} \rightarrow \mathbf{X}$ of \mathcal{C}^{op} -diagrams in \mathcal{M}^{op} at α .*

Proof. We will prove part (1); part (2) then follows by applying part (1) to the map of \mathcal{C}^{op} -diagrams $f^{\text{op}}: \mathbf{Y} \rightarrow \mathbf{X}$ in \mathcal{M}^{op} and remembering that $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$ and $(\mathcal{M}^{\text{op}})^{\text{op}} = \mathcal{M}$.

If $P = \mathbf{X}_\alpha \amalg_{L_\alpha^{\mathcal{C}}\mathbf{X}} L_\alpha^{\mathcal{C}}\mathbf{Y}$, then the relative latching map is the unique map $P \rightarrow \mathbf{Y}_\alpha$ that makes the diagram

$$\begin{array}{ccc} L_\alpha^{\mathcal{C}}\mathbf{X} & \longrightarrow & L_\alpha^{\mathcal{C}}\mathbf{Y} \\ \downarrow & \nearrow P & \downarrow \\ \mathbf{X}_\alpha & \longrightarrow & \mathbf{Y}_\alpha \end{array}$$

commute. The opposite of that diagram is the diagram

$$\begin{array}{ccc} M_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{X} & \longleftarrow & M_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{Y} \\ \uparrow & \nwarrow P & \uparrow \\ \mathbf{X}_\alpha & \longleftarrow & \mathbf{Y}_\alpha \end{array}$$

in \mathcal{M}^{op} (see Lemma 2.28), in which $P = \mathbf{X}_\alpha \times_{M_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{X}} M_\alpha^{\mathcal{C}^{\text{op}}}\mathbf{Y}$, and the opposite of the relative latching map is the unique map in \mathcal{M}^{op} that makes this diagram commute, i.e., it is the relative matching map. \square

Proposition 2.30. *If \mathcal{M} is a model category and \mathcal{C} is a Reedy category, then the opposite $(\mathcal{M}^{\mathcal{C}})^{\text{op}}$ of the Reedy model category $\mathcal{M}^{\mathcal{C}}$ (see Definition 2.3) is naturally isomorphic as a model category to the Reedy model category $(\mathcal{M}^{\text{op}})^{\mathcal{C}^{\text{op}}}$.*

Proof. The opposite $(\mathcal{M}^c)^{\text{op}}$ of \mathcal{M}^c is a model category in which

- the cofibrations of $(\mathcal{M}^c)^{\text{op}}$ are the opposites of the fibrations of \mathcal{M}^c ,
- the fibrations of $(\mathcal{M}^c)^{\text{op}}$ are the opposites of the cofibrations of \mathcal{M}^c , and
- the weak equivalences of $(\mathcal{M}^c)^{\text{op}}$ are the opposites of the weak equivalences of \mathcal{M}^c .

Proposition 2.24 implies that we have a Reedy model category structure on $(\mathcal{M}^{\text{op}})^{c^{\text{op}}}$. The objects and maps of $(\mathcal{M}^c)^{\text{op}}$ coincide with those of $(\mathcal{M}^{\text{op}})^{(c^{\text{op}})}$, and so we need only show that the model category structures coincide. This follows because the opposites of the objectwise weak equivalences of \mathcal{M}^c are the objectwise weak equivalences of $(\mathcal{M}^{\text{op}})^{c^{\text{op}}}$, and Lemma 2.29 implies that the opposites of the cofibrations of \mathcal{M}^c are the fibrations of $(\mathcal{M}^{\text{op}})^{c^{\text{op}}}$ and that the opposites of the fibrations of \mathcal{M}^c are the cofibrations of $(\mathcal{M}^{\text{op}})^{c^{\text{op}}}$ (see Theorem 2.5). \square

2.5. Quillen functors.

Definition 2.31. Let \mathcal{M} and \mathcal{N} be model categories and let $G: \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a pair of adjoint functors. The functor G is a *left Quillen functor* and the functor U is a *right Quillen functor* if

- the left adjoint G preserves both cofibrations and trivial cofibrations, and
- the right adjoint U preserves both fibrations and trivial fibrations.

Proposition 2.32. If \mathcal{M} and \mathcal{N} are model categories and $G: \mathcal{M} \rightleftarrows \mathcal{N} : U$ is a pair of adjoint functors, then the following are equivalent:

- (1) The left adjoint G is a left Quillen functor and the right adjoint U is a right Quillen functor.
- (2) The left adjoint G preserves both cofibrations and trivial cofibrations.
- (3) The right adjoint U preserves both fibrations and trivial fibrations.

Proof. This is [H1, Prop. 8.5.3]. \square

Proposition 2.33. Let \mathcal{M} and \mathcal{N} be model categories and let $G: \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a pair of adjoint functors.

- (1) If G is a left Quillen functor, then G takes cofibrant objects of \mathcal{M} to cofibrant objects of \mathcal{N} and takes weak equivalences between cofibrant objects in \mathcal{M} to weak equivalences between cofibrant objects of \mathcal{N} .
- (2) If U is a right Quillen functor, then U takes fibrant objects of \mathcal{N} to fibrant objects of \mathcal{M} and takes weak equivalences between fibrant objects in \mathcal{N} to weak equivalences between fibrant objects of \mathcal{M} .

Proof. Since left adjoints take initial objects to initial objects, if the left adjoint G takes cofibrations to cofibrations then it takes cofibrant objects to cofibrant objects. The statement about weak equivalences follows from [H1, Cor. 7.7.2].

Dually, since right adjoints take terminal objects to terminal objects, if the right adjoint U takes fibrations to fibrations then it takes fibrant objects to fibrant objects. The statement about weak equivalences follows from [H1, Cor. 7.7.2]. \square

Proposition 2.34. A functor between model categories $G: \mathcal{M} \rightarrow \mathcal{N}$ is a left Quillen functor if and only if its opposite $G^{\text{op}}: \mathcal{M}^{\text{op}} \rightarrow \mathcal{N}^{\text{op}}$ is a right Quillen functor.

Proof. This follows because the cofibrations and trivial cofibrations of \mathcal{M}^{op} are the opposites of the fibrations and trivial fibrations, respectively, of \mathcal{M} and the

fibrations and trivial fibrations of \mathcal{M}^{op} are the opposites of the cofibrations and trivial cofibrations, respectively, of \mathcal{M} (with a similar statement for \mathcal{N}). \square

2.6. Cofinality.

Definition 2.35. Let \mathcal{C} and \mathcal{D} be small categories and let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- The functor G is *left cofinal* (or *initial*) if for every object α of \mathcal{D} the nerve $N(G \downarrow \alpha)$ of the overcategory $(G \downarrow \alpha)$ is non-empty and connected. If in addition G is the inclusion of a subcategory, then we will say that \mathcal{C} is a *left cofinal subcategory* (or *initial subcategory*) of \mathcal{D} .
- The functor G is *right cofinal* (or *terminal*) if for every object α of \mathcal{D} the nerve $N(\alpha \downarrow G)$ of the undercategory $(\alpha \downarrow G)$ is non-empty and connected. If in addition G is the inclusion of a subcategory, then we will say that \mathcal{C} is a *right cofinal subcategory* (or *terminal subcategory*) of \mathcal{D} .

For the proof of the following, see [H1, Thm. 14.2.5].

Theorem 2.36. Let \mathcal{C} and \mathcal{D} be small categories and let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- (1) The functor G is left cofinal if and only if for every complete category \mathcal{M} (i.e. a category where all small limits exist) and every diagram $\mathbf{X}: \mathcal{D} \rightarrow \mathcal{M}$ the natural map $\lim_{\mathcal{D}} \mathbf{X} \rightarrow \lim_{\mathcal{C}} G^* \mathbf{X}$ is an isomorphism.
- (2) The functor G is right cofinal if and only if for every cocomplete category \mathcal{M} (i.e. a category where all small colimits exist) and every diagram $\mathbf{X}: \mathcal{D} \rightarrow \mathcal{M}$ the natural map $\text{colim}_{\mathcal{C}} G^* \mathbf{X} \rightarrow \text{colim}_{\mathcal{D}} \mathbf{X}$ is an isomorphism.

3. EXAMPLES

In this section, we present various examples to illustrate Theorem 1.1 and Theorem 1.2.

3.1. A Reedy functor that is not fibering. The following is an example of a Reedy subcategory of a square that is not fibering.

Example 3.1. Let \mathcal{D} be the category

$$\begin{array}{ccc} & \alpha & \\ p \swarrow & & \searrow r \\ \gamma & & \delta \\ q \swarrow & & \searrow s \\ & \beta & \end{array} \quad \text{in which } qp = sr.$$

- Let α be of degree 2,
- let γ and δ be of degree 1, and
- let β be of degree 0.

\mathcal{D} is then a Reedy category in which $\overleftarrow{\mathcal{D}} = \mathcal{D}$ and $\overrightarrow{\mathcal{D}}$ has only identity maps.

Let \mathcal{C} be the full subcategory of \mathcal{D} on the objects $\{\alpha, \gamma, \delta\}$, and let \mathcal{C} have the structure of a Reedy category that makes it a Reedy subcategory of \mathcal{D} . Although \mathcal{C} is a Reedy subcategory of \mathcal{D} , it is not a fibering Reedy subcategory because the map $qp: \alpha \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ has only two factorizations in which the first map is in \mathcal{C} and is not an identity map and the second is in $\overleftarrow{\mathcal{D}}$, $q \circ p$ and $s \circ r$, and neither of those factorizations maps to the other; thus the nerve of the category of such factorizations is nonempty and not connected. Theorem 1.1 thus implies that there is a model category \mathcal{M} such that the restriction functor $\mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ is not a right Quillen functor.

3.2. Truncations.

Definition 3.2. Let \mathcal{C} be a Reedy category and let $n \geq 0$.

- (1) The category $\mathcal{C}^{\leq n}$, the n 'th truncation of \mathcal{C} , is the full subcategory of \mathcal{C} with objects equal to the objects of \mathcal{C} of degree at most n .
- (2) If \mathcal{M} is a model category and \mathbf{X} is a \mathcal{C} -diagram in \mathcal{M} , then $\mathbf{X}^{\leq n}$ is the $\mathcal{C}^{\leq n}$ -diagram in \mathcal{M} that is the restriction of \mathbf{X} .

The following is a direct consequence of the definitions.

Proposition 3.3. *If \mathcal{C} is a Reedy category and $n \geq 0$, then $\mathcal{C}^{\leq n}$ is a Reedy category with $\overrightarrow{\mathcal{C}^{\leq n}} = \overrightarrow{\mathcal{C}} \cap (\mathcal{C}^{\leq n})$ and $\overleftarrow{\mathcal{C}^{\leq n}} = \overleftarrow{\mathcal{C}} \cap (\mathcal{C}^{\leq n})$*

Proposition 3.4. *If \mathcal{C} is a Reedy category and $n \geq 0$, then the inclusion functor $G: \mathcal{C}^{\leq n} \rightarrow \mathcal{C}$ is both a fibering Reedy functor and a cofibering Reedy functor.*

Proof. We will prove that the inclusion is a fibering Reedy functor; the proof that it is a cofibering Reedy functor is similar.

If $\text{degree}(\alpha) \leq n$, then the inclusion functor $G: \mathcal{C}^{\leq n} \rightarrow \mathcal{C}$ induces an isomorphism of undercategories $G_*: (\alpha \downarrow \overleftarrow{\mathcal{C}^{\leq n}}) \rightarrow (\alpha \downarrow \overleftarrow{\mathcal{C}})$. Let $\sigma: \alpha \rightarrow \beta$ be a map in $\overleftarrow{\mathcal{C}}$. If σ is the identity map, then the category of inverse \mathcal{C} -factorizations of σ is empty; if σ is not an identity map, then the object $((\sigma: \alpha \rightarrow \beta), 1_\beta)$ is a terminal object of the category of inverse \mathcal{C} -factorizations of σ , and so the nerve of the category of inverse \mathcal{C} -factorizations of σ is connected. Thus, G is fibering. \square

Proposition 3.5. *If \mathcal{M} is a model category, \mathcal{C} is a Reedy category, and $n \geq 0$, then the restriction functor $\mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{C}^{\leq n}}$ is both a left Quillen functor and a right Quillen functor.*

Proof. This follows from Proposition 3.4, Theorem 1.1, and Theorem 1.2. \square

Proposition 3.5 extends to products of Reedy categories as follows.

Proposition 3.6. *If \mathcal{C} and \mathcal{D} are Reedy categories, \mathcal{M} is a model category, and $n \geq 0$, then the restriction functor $\mathcal{M}^{\mathcal{C} \times \mathcal{D}} \rightarrow \mathcal{M}^{(\mathcal{C}^{\leq n} \times \mathcal{D})}$ is both a left Quillen functor and a right Quillen functor.*

Proof. The category $\mathcal{M}^{\mathcal{C} \times \mathcal{D}}$ of $(\mathcal{C} \times \mathcal{D})$ -diagrams in \mathcal{M} is isomorphic as a model category to the category $(\mathcal{M}^{\mathcal{D}})^{\mathcal{C}}$ of \mathcal{C} -diagrams in $\mathcal{M}^{\mathcal{D}}$ (see [H1, Thm. 15.5.2]), and so the result follows from Proposition 3.5. \square

Proposition 3.7. *If \mathcal{M} is a model category, m is a positive integer, and for $1 \leq i \leq m$ we have a Reedy category \mathcal{C}_i and a nonnegative integer n_i , then the restriction functor*

$$\mathcal{M}^{\mathcal{C}_1 \times \mathcal{C}_2 \times \dots \times \mathcal{C}_m} \longrightarrow \mathcal{M}^{\mathcal{C}_1^{\leq n_1} \times \mathcal{C}_2^{\leq n_2} \times \dots \times \mathcal{C}_m^{\leq n_m}}$$

is both a left Quillen functor and a right Quillen functor.

Proof. The restriction functor is the composition of the restriction functors

$$\begin{aligned} \mathcal{M}^{\mathcal{C}_1 \times \mathcal{C}_2 \times \dots \times \mathcal{C}_m} &\longrightarrow \mathcal{M}^{\mathcal{C}_1^{\leq n_1} \times \mathcal{C}_2 \times \dots \times \mathcal{C}_m} \\ &\longrightarrow \mathcal{M}^{\mathcal{C}_1^{\leq n_1} \times \mathcal{C}_2^{\leq n_2} \times \dots \times \mathcal{C}_m} \longrightarrow \dots \longrightarrow \mathcal{M}^{\mathcal{C}_1^{\leq n_1} \times \mathcal{C}_2^{\leq n_2} \times \dots \times \mathcal{C}_m^{\leq n_m}} \end{aligned}$$

and so the result follows from Proposition 3.6. \square

3.3. Skeleta.

Definition 3.8. Let \mathcal{C} be a Reedy category, let $n \geq 0$, and let \mathcal{M} be a model category.

- (1) Since \mathcal{M} is cocomplete, the restriction functor $\mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{e^{\leq n}}$ has a left adjoint $\mathbf{L}: \mathcal{M}^{e^{\leq n}} \rightarrow \mathcal{M}^{\mathcal{C}}$ (see [B, Thm. 3.7.2]), and we define the *n-skeleton functor* $\mathrm{sk}_n: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{C}}$ to be the composition

$$\mathcal{M}^{\mathcal{C}} \xrightarrow{\text{restriction}} \mathcal{M}^{e^{\leq n}} \xrightarrow{\mathbf{L}} \mathcal{M}^{\mathcal{C}}.$$

- (2) Since \mathcal{M} is complete, the restriction functor $\mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{e^{\leq n}}$ has a right adjoint $\mathbf{R}: \mathcal{M}^{e^{\leq n}} \rightarrow \mathcal{M}^{\mathcal{C}}$ (see [B, Thm. 3.7.2]), and we define the *n-coskeleton functor* $\mathrm{cosk}_n: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{C}}$ to be the composition

$$\mathcal{M}^{\mathcal{C}} \xrightarrow{\text{restriction}} \mathcal{M}^{e^{\leq n}} \xrightarrow{\mathbf{R}} \mathcal{M}^{\mathcal{C}}.$$

Proposition 3.9. If \mathcal{C} is a Reedy category, $n \geq 0$, and \mathcal{M} is a model category, then

- (1) the *n-skeleton functor* $\mathrm{sk}_n: \mathcal{M} \rightarrow \mathcal{M}$ is a left Quillen functor, and
- (2) the *n-coskeleton functor* $\mathrm{cosk}_n: \mathcal{M} \rightarrow \mathcal{M}$ is a right Quillen functor.

Proof. Since the restriction functor is a right Quillen functor (see Proposition 3.5), its left adjoint is a left Quillen functor (see Proposition 2.32). Since the restriction is also a left Quillen functor (see Proposition 3.5), its composition with its left adjoint is a left Quillen functor. Similarly, the composition of restriction with its right adjoint is a right Quillen functor. \square

3.4. (Multi)cosimplicial and (multi)simplicial objects. In this section we consider simplicial and cosimplicial diagrams as well as their multidimensional versions (see Definition 3.10). Simplicial and cosimplicial diagrams are standard tools in homotopy theory, while *m*-simplicial and *m*-cosimplicial ones have seen an increase in usage in recent years, most notably through their appearance in the calculus of functors (see [E, KMV]).

The important questions are whether the restrictions to various subdiagrams of *m*-simplicial and *m*-cosimplicial diagrams are Quillen functors (and the answer will be yes in all cases). The subdiagrams we will look at are the restricted (co)simplicial objects, diagonals of *m*-(co)simplicial objects, and slices of *m*-(co)simplicial objects. These are considered in Sections 3.4.1, 3.4.2, and 3.4.3, respectively. In particular, the fibrancy of the slices of a fibrant *m*-dimensional cosimplicial object is needed to justify taking its totalization one dimension at a time, as is done in both [E] and [KMV]. This and some further results about totalizations of *m*-cosimplicial objects will be addressed in future work.

Definition 3.10. For n a nonnegative integer, let $[n]$ denote the ordered set $(0, 1, 2, \dots, n)$.

- (1) The *cosimplicial indexing category* Δ is the category with objects the $[n]$ for $n \geq 0$ and with $\Delta([n], [k])$ the set of weakly monotone functions $[n] \rightarrow [k]$.
- (2) A *cosimplicial object* in a category \mathcal{M} is a functor from Δ to \mathcal{M} .
- (3) If m is a positive integer, then an *m-cosimplicial object* in \mathcal{M} is a functor from Δ^m to \mathcal{M} .
- (4) The *simplicial indexing category* Δ^{op} , the opposite category of Δ .
- (5) A *simplicial object* in a category \mathcal{M} is a functor from Δ^{op} to \mathcal{M} .

- (6) If m is a positive integer, then an m -simplicial object in \mathcal{M} is a functor from $(\Delta^m)^{\text{op}} = (\Delta^{\text{op}})^m$ to \mathcal{M} .

3.4.1. *Restricted cosimplicial objects and restricted simplicial objects.* For examples of fibering Reedy subcategories and cofiber Reedy subcategories that include all of the objects, we consider the restricted cosimplicial (or semi-cosimplicial) and restricted simplicial (or semi-simplicial) indexing categories.

Definition 3.11. For n a nonnegative integer, let $[n]$ denote the ordered set $(0, 1, 2, \dots, n)$.

- (1) The *restricted cosimplicial indexing category* Δ_{rest} is the category with objects the ordered sets $[n]$ for $n \geq 0$ and with $\Delta_{\text{rest}}([n], [k])$ the *injective* order preserving maps $[n] \rightarrow [k]$.
The category Δ_{rest} is thus a subcategory of Δ , the cosimplicial indexing category (see Definition 3.10).
- (2) The *restricted simplicial indexing category* $\Delta_{\text{rest}}^{\text{op}}$ is the opposite of the restricted cosimplicial indexing category.
- (3) If \mathcal{M} is a category, then a *restricted cosimplicial object* in \mathcal{M} is a functor from Δ_{rest} to \mathcal{M} .
- (4) If \mathcal{M} is a category, a *restricted simplicial object* in \mathcal{M} is a functor from $(\Delta_{\text{rest}})^{\text{op}}$ to \mathcal{M} .

If we let $G: \Delta_{\text{rest}} \rightarrow \Delta$ be the inclusion, then for \mathbf{X} a cosimplicial object in \mathcal{M} the induced diagram $G^*\mathbf{X}$ is a restricted cosimplicial object in \mathcal{M} , called the *underlying restricted cosimplicial object* of \mathbf{X} ; it is obtained from \mathbf{X} by “forgetting the codegeneracy operators”. Similarly, if we let $G: \Delta_{\text{rest}}^{\text{op}} \rightarrow \Delta^{\text{op}}$ be the inclusion, then for \mathbf{Y} a simplicial object in \mathcal{M} the induced diagram $G^*\mathbf{Y}$ is a restricted simplicial object in \mathcal{M} , called the *underlying restricted simplicial object* of \mathbf{Y} , obtained from \mathbf{Y} by “forgetting the degeneracy operators”.

Theorem 3.12.

- (1) The inclusion $\Delta_{\text{rest}} \rightarrow \Delta$ of the restricted cosimplicial indexing category into the cosimplicial indexing category is both a fibering Reedy functor and a cofiber Reedy functor.
- (2) The inclusion $\Delta_{\text{rest}}^{\text{op}} \rightarrow \Delta^{\text{op}}$ of the restricted simplicial indexing category into the simplicial indexing category is both a fibering Reedy functor and a cofiber Reedy functor.

Proof. We will prove part 1; part 2 will then follow from Proposition 2.27.

We first prove that the inclusion $\Delta_{\text{rest}} \rightarrow \Delta$ is the inclusion of a cofiber Reedy subcategory. Let $\sigma: \beta \rightarrow \alpha$ be a map in $\overrightarrow{\Delta}$. If σ is an identity map, then the category of direct Δ_{rest} -factorizations of σ is empty. If σ is not an identity map, then $((\sigma: \beta \rightarrow \alpha), 1_\beta)$ is an object of the category of direct Δ_{rest} -factorizations of σ that maps to every other object of that category, and so the nerve of that category is connected.

We now prove that the inclusion $\Delta_{\text{rest}} \rightarrow \Delta$ is the inclusion of a fibering Reedy subcategory. Let $\sigma: \alpha \rightarrow \beta$ be a map in $\overleftarrow{\Delta}$. Since there are no non-identity maps in $\overleftarrow{\Delta_{\text{rest}}}$, the category of inverse Δ_{rest} -factorizations of σ is empty. \square

Theorem 3.13. Let \mathcal{M} be a model category.

- (1) The functor $\mathcal{M}^{\Delta} \rightarrow \mathcal{M}^{\Delta_{\text{rest}}}$ that “forgets the codegeneracies” of a cosimplicial object is both a left Quillen functor and a right Quillen functor.
- (2) The functor $\mathcal{M}^{\Delta^{\text{op}}} \rightarrow \mathcal{M}^{\Delta_{\text{rest}}^{\text{op}}}$ that “forgets the degeneracies” of a simplicial object is both a left Quillen functor and a right Quillen functor.

Proof. This follows from Theorem 3.12, Theorem 1.1, and Theorem 1.2. \square

3.4.2. Diagonals of multicosimplicial and multisimplicial objects.

Definition 3.14. Let m be a positive integer.

- (1) The *diagonal embedding* of the category Δ into Δ^m is the functor $D: \Delta \rightarrow \Delta^m$ that takes the object $[k]$ of Δ to the object $\underbrace{([k], [k], \dots, [k])}_{m \text{ times}}$ of Δ^m and the morphism $\phi: [p] \rightarrow [q]$ of Δ to the morphism $\underbrace{(\phi^m)}_{m \text{ times}}$ of Δ^m .
- (2) If \mathcal{M} is a category and \mathbf{X} is an m -cosimplicial object in \mathcal{M} , then the *diagonal* $\text{diag } \mathbf{X}$ of \mathbf{X} is the cosimplicial object in \mathcal{M} that is the composition

$$\Delta \xrightarrow{D} \Delta^m \xrightarrow{\mathbf{X}} \mathcal{M},$$

so that $(\text{diag } \mathbf{X})^k = \mathbf{X}^{(k, k, \dots, k)}$.

- (3) If \mathcal{M} is a category and \mathbf{X} is an m -simplicial object in \mathcal{M} , then the *diagonal* $\text{diag } \mathbf{X}$ of \mathbf{X} is the simplicial object in \mathcal{M} that is the composition

$$\Delta^{\text{op}} \xrightarrow{D^{\text{op}}} (\Delta^m)^{\text{op}} = (\Delta^{\text{op}})^m \xrightarrow{\mathbf{X}} \mathcal{M},$$

so that $(\text{diag } \mathbf{X})_k = \mathbf{X}_{(k, k, \dots, k)}$.

Theorem 3.15. Let m be a positive integer.

- (1) The diagonal embedding $D: \Delta \rightarrow \Delta^m$ is a fibering Reedy functor.
- (2) The diagonal embedding $D^{\text{op}}: \Delta^{\text{op}} \rightarrow (\Delta^m)^{\text{op}} = (\Delta^{\text{op}})^m$ is a cofibering Reedy functor.

Proof. We will prove part 1; part 2 will then follow from Proposition 2.27.

We will identify Δ with its image in Δ^m , so that the objects of Δ are the m -tuples $([k], [k], \dots, [k])$. If $(\alpha_1, \alpha_2, \dots, \alpha_m): ([k], [k], \dots, [k]) \rightarrow ([p_1], [p_2], \dots, [p_m])$ is a map in Δ^m , then [H2, Lem. 5.1] implies that it has a terminal factorization through a diagonal object of Δ^m . If that terminal factorization is through the identity map of $([k], [k], \dots, [k])$, then the category of inverse Δ -factorizations of $(\alpha_1, \alpha_2, \dots, \alpha_m)$ is empty; if that terminal factorization is not through the identity map, then it is a terminal object of the category of inverse Δ -factorizations of $(\alpha_1, \alpha_2, \dots, \alpha_m)$, and so the nerve of that category is connected. \square

Part 1 of the following corollary appears in [H2].

Corollary 3.16. Let m be a positive integer and let \mathcal{M} be a model category.

- (1) The functor that takes an m -cosimplicial object in \mathcal{M} to its diagonal cosimplicial object is a right Quillen functor.
- (2) The functor that takes an m -simplicial object in \mathcal{M} to its diagonal simplicial object is a left Quillen functor.

Proof. This follows from Theorem 3.15, Theorem 1.1, and Theorem 1.2. \square

3.4.3. Slices of multicosimplicial and multisimplicial objects.

Definition 3.17. Let n be a positive integer and for $1 \leq i \leq n$ let \mathcal{C}_i be a category. If K is a subset of $\{1, 2, \dots, n\}$, then a K -slice of the product category $\prod_{i=1}^n \mathcal{C}_i$ is the category $\prod_{i \in K} \mathcal{C}_i$. (If K consists of a single integer j , then we will use the term j -slice to refer to the K -slice.) An *inclusion of a K -slice* is a functor $\prod_{i \in K} \mathcal{C}_i \rightarrow \prod_{i=1}^n \mathcal{C}_i$ defined by choosing an object α_i of \mathcal{C}_i for $i \in (\{1, 2, \dots, n\} - K)$ and inserting α_i into the i 'th coordinate for $i \in (\{1, 2, \dots, n\} - K)$.

Theorem 3.18. Let n be a positive integer and for $1 \leq i \leq n$ let \mathcal{C}_i be a Reedy category. For every subset K of $\{1, 2, \dots, n\}$ both the product $\prod_{i=1}^n \mathcal{C}_i$ and the product $\prod_{i \in K} \mathcal{C}_i$ are Reedy categories (see [H1, Prop. 15.1.6]), and every inclusion of a K -slice $\prod_{i \in K} \mathcal{C}_i \rightarrow \prod_{i=1}^n \mathcal{C}_i$ (see Definition 3.17) is both a fibering Reedy functor and a cofibering Reedy functor.

Proof. We will show that every inclusion is a fibering Reedy functor; the proof that it is a cofibering Reedy functor is similar (and also follows from applying the fibering case to the inclusion $\prod_{i \in K} \mathcal{C}_i^{\text{op}} \rightarrow \prod_{i=1}^n \mathcal{C}_i^{\text{op}}$; see Proposition 2.27). We will assume that $K = \{1, 2\}$; the other cases are similar.

Let $(\beta_1, \beta_2, \alpha_3, \alpha_4, \dots, \alpha_n)$ be an object of $\prod_{i \in K} \mathcal{C}_i$ and let

$$(\sigma_1, \sigma_2, \dots, \sigma_n): (\beta_1, \beta_2, \alpha_3, \alpha_4, \dots, \alpha_n) \longrightarrow (\gamma_1, \gamma_2, \dots, \gamma_n)$$

be a map in $\overleftarrow{\prod_{i=1}^n \mathcal{C}_i}$. Since $\overleftarrow{\prod_{i=1}^n \mathcal{C}_i} = \prod_{i=1}^n \overleftarrow{\mathcal{C}_i}$, each $\sigma_i \in \overleftarrow{\mathcal{C}_i}$. If σ_1 and σ_2 are both identity maps, then the category of inverse $\prod_{i \in K} \mathcal{C}_i$ -factorizations of $(\sigma_1, \sigma_2, \dots, \sigma_n)$ is empty. Otherwise, the category of inverse $\prod_{i \in K} \mathcal{C}_i$ -factorizations of $(\sigma_1, \sigma_2, \dots, \sigma_n)$ contains the object

$$\begin{aligned} (\beta_1, \beta_2, \alpha_3, \alpha_4, \dots, \alpha_n) &\xrightarrow{(\sigma_1, \sigma_2, 1e_3, 1e_4, \dots, 1e_n)} (\gamma_1, \gamma_2, \alpha_3, \alpha_4, \dots, \alpha_n) \\ &\xrightarrow{(1e_1, 1e_2, \sigma_3, \sigma_4, \dots, \sigma_n)} (\gamma_1, \gamma_2, \dots, \gamma_n) \end{aligned}$$

and every other object of the category of inverse $\prod_{i \in K} \mathcal{C}_i$ -factorizations of $(\sigma_1, \sigma_2, \dots, \sigma_n)$ maps to this one. Thus the nerve of the category of inverse $\prod_{i \in K} \mathcal{C}_i$ -factorizations of $(\sigma_1, \sigma_2, \dots, \sigma_n)$ is connected. \square

Theorem 3.19. If \mathcal{M} is a model category and n, \mathcal{C}_i for $1 \leq i \leq n$, and K are as in Theorem 3.18 and the functor $\prod_{i \in K} \mathcal{C}_i \rightarrow \prod_{i=1}^n \mathcal{C}_i$ is the inclusion of a K -slice, then the restriction functor

$$\mathcal{M}(\prod_{i=1}^n \mathcal{C}_i) \longrightarrow \mathcal{M}(\prod_{i \in K} \mathcal{C}_i)$$

is both a left Quillen functor and a right Quillen functor.

Proof. The functor $\mathcal{M}(\prod_{i=1}^n \mathcal{C}_i) \rightarrow \mathcal{M}(\prod_{i \in K} \mathcal{C}_i)$ is induced by the functor $G: \prod_{i \in K} \mathcal{C}_i \rightarrow \prod_{i=1}^n \mathcal{C}_i$ that uses the α_i for all coordinates in $(\{1, 2, \dots, n\} - K)$, and so Theorem 1.1 and Theorem 1.2 imply that it is sufficient to show that G is the inclusion of a subcategory that is both a fibering Reedy subcategory and a cofibering Reedy subcategory. That is the statement of Theorem 3.18. \square

Definition 3.20. Let \mathcal{M} be a model category and let m be a positive integer.

- (1) If \mathbf{X} is an m -cosimplicial object in \mathcal{M} , then a *slice* of \mathbf{X} is a cosimplicial object in \mathcal{M} defined by restricting all but one factor of Δ^m .

- (2) If \mathbf{X} is an m -simplicial object in \mathcal{M} , then a *slice* of \mathbf{X} is a simplicial object in \mathcal{M} defined by restricting all but one factor of $(\Delta^{\text{op}})^m$.

Theorem 3.21. *Let \mathcal{M} be a model category and let m be a positive integer.*

- (1) *If \mathcal{C} is a slice of the multicosimplicial indexing category Δ^m , then the functor $\mathcal{M}^{\Delta^m} \rightarrow \mathcal{M}^{\Delta}$ that restricts a multicosimplicial object to the slice \mathcal{C} is both a left Quillen functor and a right Quillen functor.*
- (2) *If \mathcal{C} is a slice of the multisimplicial indexing category $(\Delta^{\text{op}})^m$, then the functor $\mathcal{M}^{(\Delta^{\text{op}})^m} \rightarrow \mathcal{M}^{\Delta^{\text{op}}}$ that restricts a multisimplicial object to the slice \mathcal{C} is both a left Quillen functor and a right Quillen functor.*

Proof. This follows from Theorem 3.19. □

Corollary 3.22. *Let \mathcal{M} be a model category and let m be a positive integer.*

- (1) *If \mathbf{X} is a fibrant m -cosimplicial object in \mathcal{M} , then every slice of \mathbf{X} is a fibrant cosimplicial object.*
- (2) *If \mathbf{X} is a cofibrant m -simplicial object in \mathcal{M} , then every slice of \mathbf{X} is a cofibrant simplicial object.*

Proof. This follows from Theorem 3.21. □

4. PROOFS OF THE MAIN THEOREMS

Our main result, Theorem 1.1, will follow from Theorem 4.1 below. The proof of its dual, Theorem 1.2, will use Theorem 1.1 and can be found in Section 4.3.

Theorem 4.1. *If $G: \mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor between Reedy categories, then the following are equivalent:*

- (1) *The functor G is a fibering Reedy functor (see Definition 2.16).*
- (2) *For every model category \mathcal{M} the induced functor of diagram categories $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ is a right Quillen functor.*
- (3) *For every model category \mathcal{M} the induced functor of diagram categories $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ takes fibrant objects of $\mathcal{M}^{\mathcal{D}}$ to fibrant objects of $\mathcal{M}^{\mathcal{C}}$.*

Theorem 4.2 shows that (1) \implies (2), Proposition 2.33 implies that (2) \implies (3) and Theorem 4.3 implies that (3) \implies (1).

Theorem 4.2. *If $G: \mathcal{C} \rightarrow \mathcal{D}$ is a fibering Reedy functor and \mathcal{M} is a model category, then the induced functor of diagram categories $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ is a right Quillen functor.*

Theorem 4.3. *If $G: \mathcal{C} \rightarrow \mathcal{D}$ is a Reedy functor that is not a fibering Reedy functor, then there is a fibrant \mathcal{D} -diagram of topological spaces for which the induced \mathcal{C} -diagram is not fibrant.*

The proof of Theorem 4.2 is given in Section 4.1, while the proof of Theorem 4.3 can be found in Section 4.2.

4.1. Proof of Theorem 4.2. We work backward, first giving the proof of the main result, and then unravelling the necessary statements that lead to it. To aid

the reader, here is the structure of the argument:

$$\begin{array}{c}
 \text{Theorem 4.2} \\
 \Updownarrow \\
 \text{Proposition 4.10} \implies \text{Proposition 4.7} \\
 \Updownarrow \\
 \text{Lemma 4.19} \implies \text{Proposition 4.11} \Leftarrow \text{Lemma 4.18} \\
 \Updownarrow \\
 \text{Lemma 4.12 \& Diagram 4.15} \implies \text{Lemma 4.16}
 \end{array}
 \tag{4.4}$$

The assumption that we have a fibering Reedy functor is used only in the proofs of Proposition 4.7 and Proposition 4.10.

Proof of Theorem 4.2. Since \mathcal{M} is cocomplete, the left adjoint of G^* exists (see [B, Thm. 3.7.2] or [M, p. 235]). Thus, to show that the induced functor $\mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ is a right Quillen functor, we need only show that it preserves fibrations and trivial fibrations (see Proposition 2.32). Since the weak equivalences in $\mathcal{M}^{\mathcal{D}}$ and $\mathcal{M}^{\mathcal{C}}$ are the objectwise ones, any weak equivalence in $\mathcal{M}^{\mathcal{D}}$ induces a weak equivalence in $\mathcal{M}^{\mathcal{C}}$. Thus, if we show that the induced functor preserves fibrations, then we will also know that it takes maps that are both fibrations and weak equivalences to maps that are both fibrations and weak equivalences, i.e., that it also preserves trivial fibrations.

To show that the induced functor $\mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ preserves fibrations, let $\mathbf{X} \rightarrow \mathbf{Y}$ be a fibration of \mathcal{D} -diagrams in \mathcal{M} ; we will let $G^*\mathbf{X}$ and $G^*\mathbf{Y}$ denote the induced diagrams on \mathcal{C} . For every object α of \mathcal{C} , the matching objects of \mathbf{X} and \mathbf{Y} at α in $\mathcal{M}^{\mathcal{C}}$ are

$$M_{\alpha}^{\mathcal{C}} G^* \mathbf{X} = \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^* \mathbf{X} \quad \text{and} \quad M_{\alpha}^{\mathcal{C}} G^* \mathbf{Y} = \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^* \mathbf{Y}$$

and we define $P_{\alpha}^{\mathcal{C}}$ by letting the diagram

$$\begin{array}{ccc}
 P_{\alpha}^{\mathcal{C}} & \xrightarrow{\quad \quad \quad} & (G^* \mathbf{Y})_{\alpha} \\
 \downarrow & & \downarrow \\
 M_{\alpha}^{\mathcal{C}} G^* \mathbf{X} & \longrightarrow & M_{\alpha}^{\mathcal{C}} G^* \mathbf{Y}
 \end{array}
 \tag{4.5}$$

be a pullback; we must show that the relative matching map $G^* \mathbf{X}_{\alpha} \rightarrow P_{\alpha}^{\mathcal{C}}$ is a fibration (see Theorem 2.5), and there are two cases:

- (1) There is a non-identity map $\alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$ that G takes to the identity map of $G\alpha$.
- (2) G takes every non-identity map $\alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$ to a non-identity map in $\overleftarrow{\mathcal{D}}$.

In the first case, Proposition 2.17 implies that the pullback Diagram 4.5 is isomorphic to

$$\begin{array}{ccc} P_\alpha^{\mathcal{C}} & \longrightarrow & (G^* \mathbf{Y})_\alpha \\ \downarrow & & \downarrow 1_{(G^* \mathbf{Y})_\alpha} \\ (G^* \mathbf{X})_\alpha & \longrightarrow & (G^* \mathbf{Y})_\alpha \end{array}$$

and so $P_\alpha^{\mathcal{C}}$ is isomorphic to $(G^* \mathbf{X})_\alpha$. Thus, the relative matching map is an isomorphism $(G^* \mathbf{X})_\alpha \rightarrow (G^* \mathbf{X})_\alpha$, and is thus a fibration.

We are left with the second case, and so we can assume that G takes every non-identity map $\alpha \rightarrow \gamma$ in $\overleftarrow{\mathcal{C}}$ to a non-identity map in $\overleftarrow{\mathcal{D}}$. In this case, G induces a functor $G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \rightarrow \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ that takes the object $f: \alpha \rightarrow \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ to the object $Gf: G\alpha \rightarrow G\gamma$ of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ (see Proposition 2.11).

The matching objects of \mathbf{X} and \mathbf{Y} at $G\alpha$ in $\mathcal{M}^{\mathcal{D}}$ are

$$M_{G\alpha}^{\mathcal{D}} \mathbf{X} = \lim_{\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \quad \text{and} \quad M_{G\alpha}^{\mathcal{D}} \mathbf{Y} = \lim_{\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})} \mathbf{Y}$$

and we define $P_{G\alpha}^{\mathcal{D}}$ by letting the diagram

$$\begin{array}{ccc} P_{G\alpha}^{\mathcal{D}} & \cdots \rightarrow & \mathbf{Y}_{G\alpha} \\ \downarrow & & \downarrow \\ M_{G\alpha}^{\mathcal{D}} \mathbf{X} & \longrightarrow & M_{G\alpha}^{\mathcal{D}} \mathbf{Y} \end{array}$$

be a pullback. The functor $G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \rightarrow \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ (see Proposition 2.11) induces natural maps

$$\begin{aligned} M_{G\alpha}^{\mathcal{D}} \mathbf{X} &= \lim_{\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \longrightarrow \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^* \mathbf{X} = M_\alpha^{\mathcal{C}} G^* \mathbf{X} \\ M_{G\alpha}^{\mathcal{D}} \mathbf{Y} &= \lim_{\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})} \mathbf{Y} \longrightarrow \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^* \mathbf{Y} = M_\alpha^{\mathcal{C}} G^* \mathbf{Y} \end{aligned}$$

and hence a natural transformation of diagrams

$$(M_{G\alpha}^{\mathcal{D}} \mathbf{X} \rightarrow M_{G\alpha}^{\mathcal{D}} \mathbf{X} \leftarrow \mathbf{Y}_{G\alpha}) \longrightarrow (M_\alpha^{\mathcal{C}} G^* \mathbf{X} \rightarrow M_\alpha^{\mathcal{C}} G^* \mathbf{X} \leftarrow (G^* \mathbf{Y})_\alpha) .$$

This in turn induces a natural map of pullbacks

$$(4.6) \quad P_{G\alpha}^{\mathcal{D}} \longrightarrow P_\alpha^{\mathcal{C}}$$

and our map $G^* \mathbf{X}_\alpha \rightarrow P_\alpha^{\mathcal{C}}$ is then the composition

$$(G^* \mathbf{X})_\alpha = \mathbf{X}_{G\alpha} \longrightarrow P_{G\alpha}^{\mathcal{D}} \longrightarrow P_\alpha^{\mathcal{C}} .$$

Since the map $\mathbf{X} \rightarrow \mathbf{Y}$ is a fibration in $\mathcal{M}^{\mathcal{D}}$, the relative matching map $\mathbf{X}_{G\alpha} \rightarrow P_{G\alpha}^{\mathcal{D}}$ is a fibration (see Theorem 2.5), and so it is sufficient to show that the natural map (4.6) is a fibration. That statement is the content of Proposition 4.7 below, which completes the proof of Theorem 4.2. \square

Proposition 4.7. *For every object α of \mathcal{C} , the map (4.6) is a fibration.*

Proof. We define a nested sequence of subcategories of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$

$$(4.8) \quad \mathcal{C}_{-1} \subset \mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_{k-1} = \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$$

by letting \mathcal{C}_i for $-1 \leq i \leq k-1$ be the full subcategory of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ with objects the union of

- the objects of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ whose target is of degree at most i , and
- the image under $G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \rightarrow \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ (see Proposition 2.11) of the objects of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$.

The functor $G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \rightarrow \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ factors through \mathcal{C}_{-1} and, since there are no objects of negative degree, this functor, which by abuse of notation we will also call $G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \rightarrow \mathcal{C}_{-1}$, maps onto the objects of \mathcal{C}_{-1} .

In fact, we claim that the functor $G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \rightarrow \mathcal{C}_{-1}$ is left cofinal (see Definition 2.35) and thus induces isomorphisms

$$\lim_{\mathcal{C}_{-1}} \mathbf{X} \approx \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^* \mathbf{X} \quad \text{and} \quad \lim_{\mathcal{C}_{-1}} \mathbf{Y} \approx \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^* \mathbf{Y}$$

(see Theorem 2.36). To see this, note that every object of \mathcal{C}_{-1} is of the form $G\sigma: G\alpha \rightarrow G\beta$ for some object $\sigma: \alpha \rightarrow \beta$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$. For every object $\sigma: \alpha \rightarrow \beta$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$, an object of the overcategory $(G_* \downarrow (G\sigma: G\alpha \rightarrow G\beta))$ is a pair

$$((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow G\beta))$$

where $\nu: \alpha \rightarrow \gamma$ is an object in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ and $\mu: G\gamma \rightarrow G\beta$ is a map in $\overleftarrow{\mathcal{D}}$ such that the triangle

$$\begin{array}{ccc} & G\alpha & \\ G\nu \swarrow & & \searrow G\sigma \\ G\gamma & \xrightarrow{\mu} & G\beta \end{array}$$

commutes, and a map

$$((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow G\beta)) \longrightarrow ((\nu': \alpha \rightarrow \gamma'), (\mu': G\gamma' \rightarrow G\beta))$$

is a map $\tau: \gamma \rightarrow \gamma'$ in $\overleftarrow{\mathcal{C}}$ that makes the diagrams

$$\begin{array}{ccc} & \alpha & \\ \nu \swarrow & & \searrow \nu' \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array} \quad \text{and} \quad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ \mu \searrow & & \swarrow \mu' \\ & \beta & \end{array}$$

commute. Thus, this overcategory is exactly the category of inverse \mathcal{C} -factorizations of $(\alpha, G\sigma)$ (see Proposition 2.15) and so (since G is a fibering Reedy functor) its nerve must be either empty or connected. Since it is not empty (it contains the vertex $(\alpha \xrightarrow{\sigma} \beta, 1_{G\beta})$), it is connected, and so $G_*: \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \rightarrow \mathcal{C}_{-1}$ is left cofinal.

The sequence of inclusions of categories (4.8) thus induces sequences of maps

$$\begin{aligned} \lim_{\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} &= \lim_{\mathcal{C}_{k-1}} \mathbf{X} \rightarrow \lim_{\mathcal{C}_{k-2}} \mathbf{X} \rightarrow \cdots \rightarrow \lim_{\mathcal{C}_0} \mathbf{X} \rightarrow \lim_{\mathcal{C}_{-1}} \mathbf{X} \approx \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^* \mathbf{X} \\ \lim_{\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})} \mathbf{Y} &= \lim_{\mathcal{C}_{k-1}} \mathbf{Y} \rightarrow \lim_{\mathcal{C}_{k-2}} \mathbf{Y} \rightarrow \cdots \rightarrow \lim_{\mathcal{C}_0} \mathbf{Y} \rightarrow \lim_{\mathcal{C}_{-1}} \mathbf{Y} \approx \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^* \mathbf{Y} . \end{aligned}$$

For $-1 \leq i \leq k-1$ we let P_i be the pullback

$$\begin{array}{ccc} P_i & \cdots \longrightarrow & Y_{G\alpha} \\ \downarrow & & \downarrow \\ \lim_{\mathcal{C}_i} \mathbf{X} & \longrightarrow & \lim_{\mathcal{C}_i} \mathbf{Y} . \end{array}$$

Since we have an evident map of diagrams

$$\left(\lim_{\mathcal{C}_{i+1}} \mathbf{X} \rightarrow \lim_{\mathcal{C}_{i+1}} \mathbf{Y} \leftarrow \mathbf{Y}_{G\alpha} \right) \longrightarrow \left(\lim_{\mathcal{C}_i} \mathbf{X} \rightarrow \lim_{\mathcal{C}_i} \mathbf{Y} \leftarrow \mathbf{Y}_{G\alpha} \right)$$

we also get an induced map $P_{i+1} \rightarrow P_i$ of pullbacks. We thus have a factorization of (4.6) as

$$P_{G\alpha}^{\mathcal{D}} = P_{k-1} \longrightarrow P_{k-2} \longrightarrow \cdots \longrightarrow P_{-1} \approx P_{\alpha}^{\mathcal{C}},$$

and we will show that the map $P_{i+1} \rightarrow P_i$ is a fibration for $-1 \leq i \leq k-2$.

The objects of \mathcal{C}_{i+1} that are not in \mathcal{C}_i are maps $G\alpha \rightarrow \beta$ where β is of degree $i+1$, and this set of maps can be divided into two subsets:

- the set S_{i+1} of maps $G\alpha \rightarrow \beta$ for which the category of inverse \mathcal{C} -factorizations of $(\alpha, G\alpha \rightarrow \beta)$ is nonempty, and
- the set T_{i+1} of maps for which the category of inverse \mathcal{C} -factorizations of $(\alpha, G\alpha \rightarrow \beta)$ is empty.

We let \mathcal{C}'_{i+1} be the full subcategory of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ with objects the union of S_{i+1} with the objects of \mathcal{C}_i , and define P'_{i+1} as the pullback

$$\begin{array}{ccc} P'_{i+1} & \cdots \cdots \cdots \rightarrow & \mathbf{Y}_{G\alpha} \\ \downarrow & & \downarrow \\ \lim_{\mathcal{C}'_{i+1}} \mathbf{X} & \longrightarrow & \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \end{array}$$

We have inclusions of categories $\mathcal{C}_i \subset \mathcal{C}'_{i+1} \subset \mathcal{C}_{i+1}$, and the maps

$$\lim_{\mathcal{C}_{i+1}} \mathbf{X} \longrightarrow \lim_{\mathcal{C}_i} \mathbf{X} \quad \text{and} \quad \lim_{\mathcal{C}_{i+1}} \mathbf{Y} \longrightarrow \lim_{\mathcal{C}_i} \mathbf{Y}$$

factor as

$$\lim_{\mathcal{C}_{i+1}} \mathbf{X} \longrightarrow \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \longrightarrow \lim_{\mathcal{C}_i} \mathbf{X} \quad \text{and} \quad \lim_{\mathcal{C}_{i+1}} \mathbf{Y} \longrightarrow \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \longrightarrow \lim_{\mathcal{C}_i} \mathbf{Y}.$$

These factorizations induce a factorization

$$(4.9) \quad P_{i+1} \longrightarrow P'_{i+1} \longrightarrow P_i$$

of the map $P_{i+1} \rightarrow P_i$. Proposition 4.10 below asserts that the map $P'_{i+1} \rightarrow P_i$ is an isomorphism and Proposition 4.11 asserts that the map $P_{i+1} \rightarrow P'_{i+1}$ is a fibration. Hence the map $P_{G\alpha}^{\mathcal{D}} \rightarrow P_{\alpha}^{\mathcal{C}}$ is a fibration as well. \square

Proposition 4.10. *For $-1 \leq i \leq k-2$, the map $P'_{i+1} \rightarrow P_i$ in (4.9) is an isomorphism.*

Proof. Let $\sigma: G\alpha \rightarrow \beta$ be an object of \mathcal{C}'_{i+1} that is not in \mathcal{C}_i . The objects of $(\mathcal{C}_i \downarrow \sigma)$ are commutative diagrams

$$\begin{array}{ccc} & G\alpha & \\ \nu \swarrow & & \searrow \sigma \\ \gamma & \xrightarrow{\mu} & \beta \end{array}$$

where $\nu: G\alpha \rightarrow \gamma$ is in \mathcal{C}_i and μ is in $\overleftarrow{\mathcal{D}}$. Since β is of degree $i+1$ and μ lowers degree, the degree of γ must be greater than $i+1$, and so the map $\nu: G\alpha \rightarrow \gamma$ must be of the form $G\nu': G\alpha \rightarrow G\gamma'$ for some map $\nu': \alpha \rightarrow \gamma'$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$. Thus, the objects of $(\mathcal{C}_i \downarrow \sigma)$ are pairs $((\nu': \alpha \rightarrow \gamma'), (\mu: G\gamma' \rightarrow \beta))$ where $\nu': \alpha \rightarrow \gamma'$ is a non-identity map of $\overleftarrow{\mathcal{C}}$, $\mu: G\gamma' \rightarrow \beta$ is in $\overleftarrow{\mathcal{D}}$, and $\mu \circ G\nu' = \sigma$, and $(\mathcal{C}_i \downarrow \sigma)$

is the category of inverse \mathcal{C} -factorizations of (α, σ) (see Proposition 2.15). Since G is a fibering Reedy functor, the nerve of the category of inverse \mathcal{C} -factorizations of (α, σ) is either empty or connected. Since it is nonempty (because $\sigma: G\alpha \rightarrow \beta$ is an element of S_{i+1}), the nerve of the overcategory $(\mathcal{C}_i \downarrow \sigma)$ is nonempty and connected, and so the inclusion $\mathcal{C}_i \subset \mathcal{C}'_{i+1}$ is left cofinal (see Definition 2.35). Thus, the maps $\lim_{\mathcal{C}'_{i+1}} \mathbf{X} \rightarrow \lim_{\mathcal{C}_i} \mathbf{X}$ and $\lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \rightarrow \lim_{\mathcal{C}_i} \mathbf{Y}$ are isomorphisms (see Theorem 2.36), and so the induced map $P'_{i+1} \rightarrow P_i$ is an isomorphism. \square

Proposition 4.11. *For $-1 \leq i \leq k-1$, the map $P_{i+1} \rightarrow P'_{i+1}$ is a fibration.*

The proof of Proposition 4.11 is more intricate; the reader might wish to refer to the chart (4.4) for its structure. Before we can present it, we will need several lemmas. For the first one, the reader should recall the definition of the sets T_i from the proof of Proposition 4.7.

Lemma 4.12. *For every \mathcal{D} -diagram \mathbf{X} in \mathcal{M} there is a natural pullback square*

$$(4.13) \quad \begin{array}{ccc} \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \longrightarrow & \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \\ \downarrow & & \downarrow \\ \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{X}_\beta & \longrightarrow & \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \lim_{\partial(\beta \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \end{array}$$

Proof. For every element $\sigma: G\alpha \rightarrow \beta$ of T_{i+1} , every object of the matching category $\partial(\beta \downarrow \overleftarrow{\mathcal{D}})$ is a map to an object of degree at most i , and so we have a functor $\partial(\beta \downarrow \overleftarrow{\mathcal{D}}) \rightarrow \mathcal{C}'_{i+1}$ that takes $\beta \rightarrow \gamma$ to the composition $G\alpha \xrightarrow{\sigma} \beta \rightarrow \gamma$; this induces the map $\lim_{\mathcal{C}'_{i+1}} \mathbf{X} \rightarrow \lim_{\partial(\beta \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X}$ that is the projection of the right hand vertical map onto the factor indexed by σ . We thus have a commutative square as in Diagram 4.13.

The objects of \mathcal{C}_{i+1} are the objects of \mathcal{C}'_{i+1} together with the elements of T_{i+1} , and so a map to $\lim_{\mathcal{C}_{i+1}} \mathbf{X}$ is determined by a map to $\lim_{\mathcal{C}'_{i+1}} \mathbf{X}$ and a map to $\prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{X}_\beta$. Since there are no non-identity maps in \mathcal{C}_{i+1} with codomain an element of T_{i+1} , and the only non-identity maps with domain an element $G\alpha \rightarrow \beta$ of T_{i+1} are the objects of the matching category $\partial(\beta \downarrow \overleftarrow{\mathcal{D}})$, maps to $\lim_{\mathcal{C}'_{i+1}} \mathbf{X}$ and to $\prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{X}_\beta$ determine a map to $\lim_{\mathcal{C}_{i+1}} \mathbf{X}$ if and only if their compositions to $\prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \lim_{\partial(\beta \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X}$ agree. Thus, the diagram is a pullback square. \square

Now define Q and R by letting the squares

$$(4.14) \quad \begin{array}{ccc} Q & \xrightarrow{\cdots} & \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \\ \downarrow & & \downarrow \\ \lim_{\mathcal{C}_{i+1}} \mathbf{Y} & \longrightarrow & \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{\cdots} & \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \lim_{\partial(\beta \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \\ \downarrow & & \downarrow \\ \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{Y}_\beta & \longrightarrow & \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \lim_{\partial(\beta \downarrow \overleftarrow{\mathcal{D}})} \mathbf{Y} \end{array}$$

be pullbacks, and consider the commutative diagram

$$(4.15) \quad \begin{array}{ccccc} \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \xrightarrow{s} & \lim_{\mathcal{C}'_{i+1}} \mathbf{X} & & \\ & \searrow a & \nearrow c & & \searrow \beta \\ & & Q & & \\ & \searrow \delta & \downarrow d & & \\ \lim_{\mathcal{C}_{i+1}} \mathbf{Y} & & & \xrightarrow{s'} & \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} \\ & \downarrow g & & \downarrow v & \\ \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{X}_\beta & \xrightarrow{t} & \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \lim_{\partial(\beta \downarrow \mathcal{D})} \mathbf{X} & & \\ & \downarrow b & \downarrow e & & \\ & & R & & \\ & \downarrow \gamma & \downarrow f & & \\ \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{Y}_\beta & \xrightarrow{t'} & \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \lim_{\partial(\beta \downarrow \mathcal{D})} \mathbf{Y} & & \\ & & \downarrow v' & & \end{array}$$

Lemma 4.12 implies that the front and back rectangles are pullbacks.

Lemma 4.16. *The square*

$$(4.17) \quad \begin{array}{ccc} \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \xrightarrow{a} & Q \\ u \downarrow & & \downarrow g \\ \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{X}_\alpha & \xrightarrow{b} & R \end{array}$$

is a pullback.

Proof. Let W be an object of \mathcal{M} and let $h: W \rightarrow \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{X}_\beta$ and $k: W \rightarrow Q$ be maps such that $gk = bh$; we will show that there is a unique map $\phi: W \rightarrow \lim_{\mathcal{C}_{i+1}} \mathbf{X}$ such that $a\phi = k$ and $u\phi = h$.

$$\begin{array}{ccc} W & \xrightarrow{k} & Q \\ & \searrow \phi & \downarrow g \\ & & \lim_{\mathcal{C}_{i+1}} \mathbf{X} \xrightarrow{a} Q \\ & \downarrow h & \downarrow u \\ & & \prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{X}_\beta \xrightarrow{b} R \end{array}$$

The map $ck: W \rightarrow \lim_{\mathcal{C}'_{i+1}} \mathbf{X}$ has the property that $v(ck) = egk = ebh = th$, and since the back rectangle of Diagram 4.15 is a pullback, the maps ck and h induce a map $\phi: W \rightarrow \lim_{\mathcal{C}_{i+1}} \mathbf{X}$ such that $u\phi = h$ and $s\phi = ck$. We must show that $a\phi = k$, and since Q is a pullback as in Diagram 4.14, this is equivalent to showing that $ca\phi = ck$ and $da\phi = dk$.

Since $ck = s\phi = ca\phi$, we need only show that $da\phi = dk$. Since the front rectangle of Diagram 4.15 is a pullback, it is sufficient to show that $s'da\phi = s'dk$ and $u'da\phi = u'dk$. For the first of those, we have

$$s'da\phi = s'\delta\phi = \beta s\phi = \beta ck = s'dk$$

and for the second, we have

$$u'da\phi = u'\delta\phi = \gamma u\phi = fbu\phi = fbh = fgk = u'dk$$

and so the map ϕ satisfies $a\phi = k$ and $u\phi = h$.

To see that ϕ is the unique such map, let $\psi: W \rightarrow \lim_{\mathcal{C}_{i+1}} \mathbf{X}$ be another map such that $a\psi = k$ and $u\psi = h$. We will show that $s\psi = s\phi$ and $u\psi = u\phi$; since the back rectangle of Diagram 4.15 is a pullback, this will imply that $\psi = \phi$.

Since $u\psi = h = u\phi$, we need only show that $s\psi = s\phi$, which follows because $s\psi = ca\psi = ck = s\phi$. \square

Lemma 4.18. *If $\mathbf{X} \rightarrow \mathbf{Y}$ is a fibration of \mathcal{D} -diagrams, then the natural map*

$$\lim_{\mathcal{C}_{i+1}} \mathbf{X} \longrightarrow Q = \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \times_{(\lim_{\mathcal{C}'_{i+1}} \mathbf{Y})} \lim_{\mathcal{C}_{i+1}} \mathbf{Y}$$

is a fibration.

Proof. Lemma 4.16 gives us the pullback square in Diagram 4.17 where Q and R are defined by the pullbacks in Diagram 4.14. Since $\mathbf{X} \rightarrow \mathbf{Y}$ is a fibration of \mathcal{D} -diagrams, the map $\prod_{(G\alpha \rightarrow \beta) \in T_{i+1}} \mathbf{X}_\beta \rightarrow R$ is a product of fibrations and is thus a fibration, and so the map $\lim_{\mathcal{C}_{i+1}} \mathbf{X} \rightarrow Q = \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \times_{(\lim_{\mathcal{C}'_{i+1}} \mathbf{Y})} \lim_{\mathcal{C}_{i+1}} \mathbf{Y}$ is a pullback of a fibration and is thus a fibration. \square

Lemma 4.19 (Reedy). *If both the front and back squares in the diagram*

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & B & \xrightarrow{f_B} & B' \\ & \searrow f_A & \downarrow & \searrow & \downarrow \\ & A' & \xrightarrow{\quad} & B' & \\ C & \xrightarrow{\quad} & D & \xrightarrow{f_D} & D' \\ & \searrow f_C & \downarrow & \searrow & \downarrow \\ & C' & \xrightarrow{\quad} & D' & \end{array}$$

are pullbacks and both $f_B: B \rightarrow B'$ and $C \rightarrow C' \times_{D'} D$ are fibrations, then $f_A: A \rightarrow A'$ is a fibration.

Proof. This is the dual of a lemma of Reedy (see [H1, Lem. 7.2.15 and Rem. 7.1.10]). \square

Proof of Proposition 4.11. We have a commutative diagram

$$\begin{array}{ccccc} P_{i+1} & \xrightarrow{\quad} & Y_{G\alpha} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \lim_{\mathcal{C}_{i+1}} \mathbf{X} & \xrightarrow{\quad} & \lim_{\mathcal{C}_{i+1}} \mathbf{Y} & & \\ & \searrow & \downarrow & \searrow & \\ & \lim_{\mathcal{C}'_{i+1}} \mathbf{X} & \xrightarrow{\quad} & \lim_{\mathcal{C}'_{i+1}} \mathbf{Y} & \end{array}$$

in which the front and back squares are pullbacks (by definition), and so Lemma 4.19 implies that it is sufficient to show that the map

$$\lim_{\mathcal{C}_{i+1}} \mathbf{X} \longrightarrow \lim_{\mathcal{C}'_{i+1}} \mathbf{X} \times_{(\lim_{\mathcal{C}'_{i+1}} \mathbf{Y})} \lim_{\mathcal{C}_{i+1}} \mathbf{Y}$$

is a fibration; that is the statement of Lemma 4.18. \square

4.2. Proof of Theorem 4.3. We will first construct the \mathcal{D} -diagram whose existence is asserted in Theorem 4.3. The proof of the theorem is then structured as follows:

$$(4.20) \quad \begin{array}{ccc} & & \text{Theorem 4.3} \\ & \nearrow & \uparrow \\ \text{Proposition 4.21} & & \text{Proposition 4.25} \\ & \nearrow & \uparrow \\ \text{Proposition 4.22} & \implies & \text{Proposition 4.24} \end{array}$$

The theorem will follow immediately from Proposition 4.21 (which asserts that the \mathcal{D} -diagram is fibrant) and Proposition 4.25 (which asserts that the induced \mathcal{C} -diagram is not fibrant).

Our \mathcal{D} -diagram \mathbf{X} will be a diagram in the standard model category of topological spaces. Throughout its construction, the reader should keep the square diagram from Example 3.1 in mind. In that example, the diagram \mathbf{X} that we construct here is the functor that sends each object in that square to the interval I with all the maps going to the identity map, and $G: \mathcal{C} \rightarrow \mathcal{D}$ is the inclusion of the diagram obtained by removing the degree zero object β from the square.

We will define the diagram \mathbf{X} inductively over the filtrations $F^n \mathcal{D}$ of \mathcal{D} (see Definition 2.7 and Proposition 2.9). To start this inductive construction, since $G: \mathcal{C} \rightarrow \mathcal{D}$ is not a fibering Reedy functor, there are objects $\alpha \in \text{Ob}(\mathcal{C})$ and $\beta \in \text{Ob}(\mathcal{D})$ and a map $\sigma: G\alpha \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ such that the nerve of the category of inverse \mathcal{C} -factorizations of (α, σ) (see Definition 2.13) is nonempty and not connected. Let n_β be the degree of β . Then we have two cases:

- If $n_\beta = 0$, we begin by letting $\mathbf{X}: F^0 \mathcal{D} \rightarrow \text{Top}$ take β to the unit interval I and all other objects of $F^0 \mathcal{D}$ to $*$ (the one-point space).
- If $n_\beta > 0$, we begin by letting $\mathbf{X}: F^{(n_\beta)-1} \mathcal{D} \rightarrow \text{Top}$ be the constant functor to $*$ (the one-point space). Then, to extend \mathbf{X} from $F^{(n_\beta)-1} \mathcal{D}$ to $F^{n_\beta} \mathcal{D}$, we let $\mathbf{X}_\beta = I$, the unit interval. We factor $L_\beta \mathbf{X} \rightarrow M_\beta \mathbf{X}$ as

$$L_\beta \mathbf{X} \longrightarrow I \longrightarrow M_\beta \mathbf{X}$$

where the first map is the constant map to $0 \in I$ and the second map is the unique map $I \rightarrow *$ (since $\mathbf{X}_\gamma = *$ is the terminal object of Top for all objects γ of degree less than n_β , that matching object is $*$). If γ is any other object of \mathcal{D} of degree n_β , we let $\mathbf{X}_\gamma = M_\gamma \mathbf{X}$ and let $L_\gamma \mathbf{X} \rightarrow \mathbf{X}_\gamma \rightarrow M_\gamma \mathbf{X}$ be the natural map followed by the identity map.

We now define $\mathbf{X}: F^n \mathcal{D} \rightarrow \text{Top}$ for $n > n_\beta$ inductively on n by letting $\mathbf{X}_\gamma = M_\gamma \mathbf{X}$ for every object γ and letting the factorization $L_\gamma \mathbf{X} \rightarrow \mathbf{X}_\gamma \rightarrow M_\gamma \mathbf{X}$ be the natural map followed by the identity map.

Proposition 4.21. *The \mathcal{D} -diagram of topological spaces \mathbf{X} is fibrant.*

Proof. The matching map at the object β of \mathcal{D} is the map $I \rightarrow *$, which is a fibration, and the matching map at every other object of \mathcal{D} is an identity map, which is also a fibration. \square

Proposition 4.22.

- (1) *For every object γ in \mathcal{D} the space \mathbf{X}_γ is homeomorphic to a product of unit intervals, one for each map $\gamma \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ (and so, for objects γ for which there are no maps $\gamma \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$, the space \mathbf{X}_γ is the empty product, and is thus equal to the terminal object, the one-point space $*$).*
- (2) *Under the isomorphisms of part 1, if $\tau: \gamma \rightarrow \delta$ is a map in $\overleftarrow{\mathcal{D}}$, then the projection of $\mathbf{X}_\tau: \mathbf{X}_\gamma \rightarrow \mathbf{X}_\delta$ onto the factor I of \mathbf{X}_δ indexed by a map $\mu: \delta \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ is the projection of \mathbf{X}_γ onto the factor I of \mathbf{X}_γ indexed by $\mu\tau: \gamma \rightarrow \beta$.*

Proof. We will use an induction on n to prove both parts of the proposition simultaneously for the restriction of \mathbf{X} to each filtration $F^n\mathcal{D}$ of \mathcal{D} . The induction is begun at $n = n_\beta$ because the only map in $F^{n_\beta}\overleftarrow{\mathcal{D}}$ to β is the identity map of β , the only object of $F^{n_\beta}\overleftarrow{\mathcal{D}}$ at which \mathbf{X} is not a single point is β , and $\mathbf{X}_\beta = I$.

Suppose now that $n > n_\beta$, the statement is true for the restriction of \mathbf{X} to $F^{n-1}\mathcal{D}$, and that γ is an object of degree n . The space \mathbf{X}_γ is defined to be the matching object $M_\gamma \mathbf{X} = \lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X}$. There is a discrete subcategory \mathcal{E}_γ of the matching category $\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})$ consisting of the maps $\gamma \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$, and so there is a projection map

$$M_\gamma \mathbf{X} = \lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \longrightarrow \lim_{\mathcal{E}_\gamma} \mathbf{X} = \prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} \mathbf{X}_\beta = \prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I.$$

We will show that that projection map $p: \lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \rightarrow \prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I$ is a homeomorphism by defining an inverse homeomorphism

$$q: \prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I \longrightarrow \lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X}.$$

We define the map q by defining its projection onto $\mathbf{X}_{(\tau: \gamma \rightarrow \delta)} = \mathbf{X}_\delta$ for each object $(\tau: \gamma \rightarrow \delta)$ of $\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})$. The induction hypothesis implies that $\mathbf{X}_\tau = \mathbf{X}_\delta$ is isomorphic to $\prod_{(\delta \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I$, and we let the projection onto the factor indexed by $\mu: \delta \rightarrow \beta$ be the projection of $\prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I$ onto the factor indexed by $\mu\tau: \gamma \rightarrow \beta$. To see that this defines a map to $\lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X}$, let $\nu: \delta \rightarrow \epsilon$ be a map from $\tau: \gamma \rightarrow \delta$ to $\nu\tau: \gamma \rightarrow \epsilon$ in $\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})$ (see Diagram 4.23). The induction hypothesis implies that the projection of the map $\mathbf{X}_\nu: \mathbf{X}_\tau = \mathbf{X}_\delta \rightarrow \mathbf{X}_{\nu\tau} = \mathbf{X}_\epsilon$ onto the factor of \mathbf{X}_ϵ indexed by $\xi: \epsilon \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ is the projection of $\mathbf{X}_\tau = \mathbf{X}_\delta$ onto the factor indexed by $\xi\nu: \delta \rightarrow \beta$.

$$(4.23) \quad \begin{array}{ccccc} & \gamma & & & \\ \tau \swarrow & & \searrow \nu\tau & & \\ \delta & \xrightarrow{\nu} & \epsilon & \xrightarrow{\xi} & \beta \end{array}$$

Thus, the projection of the composition $\prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I \rightarrow \mathbf{X}_\tau = \mathbf{X}_\delta \xrightarrow{\mathbf{X}_\nu} \mathbf{X}_{\nu\tau} = \mathbf{X}_\epsilon$ onto the factor indexed by $\xi: \epsilon \rightarrow \beta$ equals the projection of $\prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I$ onto the factor indexed by $\xi\nu\tau: \gamma \rightarrow \beta$, which equals that same projection of the map $\prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I \rightarrow \mathbf{X}_{\nu\tau: \gamma \rightarrow \epsilon} = \mathbf{X}_\epsilon$. Thus, we have defined the map q .

It is immediate from the definitions that pq is the identity map of $\prod_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I$. To see that qp is the identity map of $\lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X}$, we first note that the definitions immediately imply that the projection of qp onto each $\mathbf{X}_{(\gamma \rightarrow \beta)} = \mathbf{X}_\beta$ equals the corresponding projection of the identity map of $\lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X}$. If $\tau: \gamma \rightarrow \delta$ is any other object of $\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})$, then the induction hypothesis implies that $\mathbf{X}_\tau = \mathbf{X}_\delta$ is homeomorphic to the product $\prod_{(\delta \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} I$. Every $\mu: \delta \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ defines a map $\mu_*: (\tau: \gamma \rightarrow \delta) \rightarrow (\mu\tau: \gamma \rightarrow \beta)$ in $\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})$, and the induction hypothesis implies that the map $\mathbf{X}_\mu: \mathbf{X}_\tau = \mathbf{X}_\delta \rightarrow \mathbf{X}_{\mu\tau} = \mathbf{X}_\beta = I$ is projection onto the factor indexed by μ . Thus, for any map to $\lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X}$, its projection onto $\mathbf{X}_\tau = \mathbf{X}_\delta$ is determined by its projections onto the $\mathbf{X}_{(\gamma \rightarrow \beta) \in \overleftarrow{\mathcal{D}}}$; since qp and the identity map agree on those projections, qp must equal the identity map. This completes the induction for part 1.

For part 2, for every map $\tau: \gamma \rightarrow \delta$ in $\overleftarrow{\mathcal{D}}$ the map $\mathbf{X}_\tau: \mathbf{X}_\gamma \rightarrow \mathbf{X}_\delta$ equals the composition

$$\mathbf{X}_\gamma \longrightarrow \lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \longrightarrow \mathbf{X}_\delta$$

where the first map is the matching map of \mathbf{X} at γ and the second is the projection from the limit $\lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \rightarrow \mathbf{X}_{(\tau: \gamma \rightarrow \delta)} = \mathbf{X}_\delta$ (this is the case for every \mathcal{D} -diagram in \mathcal{M} , not just for \mathbf{X}). Since the matching map at every object other than β is the identity map, the map $\mathbf{X}_\tau: \mathbf{X}_\gamma \rightarrow \mathbf{X}_\delta$ is the projection $\lim_{\partial(\gamma \downarrow \overleftarrow{\mathcal{D}})} \mathbf{X} \rightarrow \mathbf{X}_{(\tau: \gamma \rightarrow \delta)} = \mathbf{X}_\delta$. The discussion in the previous paragraph shows that the projection of $\mathbf{X}_\tau: \mathbf{X}_\gamma \rightarrow \mathbf{X}_\delta$ onto the factor of \mathbf{X}_δ indexed by $\mu: \delta \rightarrow \beta$ is the projection of \mathbf{X}_γ onto the factor indexed by $\mu\tau: \gamma \rightarrow \beta$. This completes the induction for part 2. \square

We now consider the diagram $G^*\mathbf{X}$ that $G: \mathcal{C} \rightarrow \mathcal{D}$ induces on \mathcal{C} from \mathbf{X} .

Proposition 4.24. *The matching object $M_\alpha^{\mathcal{C}} G^*\mathbf{X} = \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} G^*\mathbf{X}$ of the induced diagram on \mathcal{C} at α is homeomorphic to a product of unit intervals indexed by the union over the maps $\tau: G\alpha \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ of the sets of path components of the nerve of the category of inverse \mathcal{C} -factorizations of (α, τ) . That is,*

$$M_\alpha^{\mathcal{C}} G^*\mathbf{X} \approx \prod_{(\tau: G\alpha \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} \left(\prod_{\pi_0 N(\text{category of inverse } \mathcal{C}\text{-factorizations of } \tau)} I \right).$$

Proof. Let $S = \coprod_{(\alpha \rightarrow \gamma) \in \text{Ob}(\partial(\alpha \downarrow \overleftarrow{\mathcal{C}}))} \overleftarrow{\mathcal{D}}(G\gamma, \beta)$, the disjoint union over all objects $\alpha \rightarrow \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ of the set of maps $\overleftarrow{\mathcal{D}}(G\gamma, \beta)$. An element of S is then an ordered pair $((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$ where $\nu: \alpha \rightarrow \gamma$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ and $\mu: G\gamma \rightarrow \beta$ is a map in $\overleftarrow{\mathcal{D}}$, and is thus an object of the category of inverse \mathcal{C} -factorizations of the composition $(\alpha, G\alpha \xrightarrow{G\nu} G\gamma \xrightarrow{\mu} \beta)$, i.e., of $(\alpha, \mu \circ G\nu: G\alpha \rightarrow \beta)$. Every object of the category of inverse \mathcal{C} -factorizations of every map $(\alpha, \tau: G\alpha \rightarrow \beta)$

in $\overleftarrow{\mathcal{D}}$ appears exactly once, and so the set S is the union over all maps $\tau: G\alpha \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ of the set of objects of the category of inverse \mathcal{C} -factorizations of (α, τ) .

Proposition 4.22 implies that for every object $\tau: \alpha \rightarrow \gamma$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ the space $(G^*\mathbf{X})_\tau = (G^*\mathbf{X})_\gamma = \mathbf{X}_{G\gamma}$ is a product of unit intervals, one for each map $G\gamma \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$, and so the product over all objects $\tau: \alpha \rightarrow \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ of $(G^*\mathbf{X})_\tau = (G^*\mathbf{X})_\gamma = \mathbf{X}_{G\gamma}$ is homeomorphic to the product of unit intervals indexed by S , i.e.,

$$\prod_{(\alpha \rightarrow \gamma) \in \text{Ob}(\partial(\alpha \downarrow \overleftarrow{\mathcal{C}}))} (G^*\mathbf{X})_\gamma \approx \prod_S I.$$

The matching object $M_\alpha^{\mathcal{C}} G^*\mathbf{X}$ is a subspace of that product. More specifically, it is the subspace consisting of the points such that, for every map

$$\begin{array}{ccc} & \alpha & \\ \nu \swarrow & & \searrow \nu' \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array}$$

in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ from $\nu: \alpha \rightarrow \gamma$ to $\nu': \alpha \rightarrow \gamma'$ and every map $\mu': G\gamma' \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$, the projection onto the factor indexed by $((\nu': \alpha \rightarrow \gamma'), (\mu': G\gamma' \rightarrow \beta))$ equals the projection onto the factor indexed by $((\nu: \alpha \rightarrow \gamma), (\mu' \circ (G\tau): G\gamma \rightarrow \beta))$.

Generate an equivalence relation on S by letting $((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$ be equivalent to $((\nu': \alpha \rightarrow \gamma'), (\mu': G\gamma' \rightarrow \beta))$ if there is a map $\tau: \gamma \rightarrow \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that $\tau\nu = \nu'$ and $\mu' \circ (G\tau) = \mu$, i.e., if there is a map in the category of inverse \mathcal{C} -factorizations of $(\alpha, \mu \circ (G\nu): G\alpha \rightarrow \beta)$ from $((\nu: \alpha \rightarrow \gamma), (\mu: G\gamma \rightarrow \beta))$ to $((\nu': \alpha \rightarrow \gamma'), (\mu': G\gamma' \rightarrow \beta))$; let T be the set of equivalence classes. This makes two objects in the category of inverse \mathcal{C} -factorizations of a map equivalent if there is a zig-zag of maps in that category from one to the other, i.e., if those two objects are in the same component of the nerve, and so the set T is the disjoint union over all maps $\tau: G\alpha \rightarrow \beta$ in $\overleftarrow{\mathcal{D}}$ of the set of components of the nerve of the category of inverse \mathcal{C} -factorizations of (α, τ) , i.e.,

$$T = \coprod_{(\tau: G\alpha \rightarrow \beta) \in \overleftarrow{\mathcal{D}}} \pi_0 N(\text{category of inverse } \mathcal{C}\text{-factorizations of } (\alpha, \tau)).$$

Let T' be a set of representatives of the equivalence classes T (i.e., let T' consist of one element of S from each equivalence class); we will show that the composition

$$M_\alpha^{\mathcal{C}} G^*\mathbf{X} \xrightarrow{\subset} \prod_S I \xrightarrow{p'} \prod_{T'} I$$

(where p' is the projection) is a homeomorphism. We will do that by constructing an inverse $q: \prod_{T'} I \rightarrow M_\alpha^{\mathcal{C}} G^*\mathbf{X}$ to the map $p: M_\alpha^{\mathcal{C}} G^*\mathbf{X} \rightarrow \prod_{T'} I$ (where p is the restriction of p' to $M_\alpha^{\mathcal{C}} G^*\mathbf{X}$).

We first construct a map $q': \prod_{T'} I \rightarrow \prod_S I$ by letting the projection of q' onto the factor indexed by $s \in S$ be the projection of $\prod_{T'} I$ onto the factor indexed by the unique $t \in T'$ that is equivalent to s . The description above of the subspace $M_\alpha^{\mathcal{C}} G^*\mathbf{X}$ of $\prod_S I$ makes it clear that q' factors through $M_\alpha^{\mathcal{C}} G^*\mathbf{X}$ and thus defines a map $q: \prod_{T'} I \rightarrow M_\alpha^{\mathcal{C}} \mathbf{X}$.

The composition pq equals the identity of $\prod_{T'} I$ because the composition $p'q'$ equals the identity of $\prod_{T'} I$. To see that the composition qp equals the identity of

$M_\alpha^{\mathcal{C}}G^*\mathbf{X}$, it is sufficient to see that the projection of qp onto the factor I indexed by every element s of S agrees with that of the identity map of $M_\alpha^{\mathcal{C}}G^*\mathbf{X}$. Since the projections of points in $M_\alpha^{\mathcal{C}}G^*\mathbf{X}$ onto factors indexed by equivalent elements of S are equal, and it is immediate that the projection of $M_\alpha^{\mathcal{C}}G^*\mathbf{X}$ onto a factor indexed by an element of the set of representatives T' agrees with the corresponding projection of qp , the projections for every element of S must agree, and so qp equals the identity of $\prod_{T'} I$. \square

Proposition 4.25. *The diagram $G^*\mathbf{X}$ induced on \mathcal{C} is not a fibrant \mathcal{C} -diagram.*

Proof. We will show that the matching map $(G^*\mathbf{X})_\alpha \rightarrow M_\alpha^{\mathcal{C}}G^*\mathbf{X}$ of the induced \mathcal{C} -diagram at α is not a fibration. Since the matching object $M_\alpha^{\mathcal{C}}G^*\mathbf{X}$ is a product of unit intervals (see Proposition 4.24), it is path connected, and so if the matching map were a fibration, it would be surjective. We will show that the matching map is not surjective.

Since $\sigma: G\alpha \rightarrow \beta$ is a map in $\overleftarrow{\mathcal{D}}$ such that the nerve of the category of inverse \mathcal{C} -factorizations of (α, σ) is not connected, we can choose objects $(\nu: \alpha \rightarrow \gamma, \mu: G\gamma \rightarrow \beta)$ and $(\nu': \alpha \rightarrow \gamma', \mu': G\gamma' \rightarrow \beta)$ of that category that represent different path components of that nerve. Since $\mu \circ (G\nu) = \mu' \circ (G\nu')$, Proposition 4.22 implies that the projection of the matching map onto the copies of I indexed by those objects are equal, and so the projection onto the $I \times I$ indexed by that pair of components factors as the composition $\mathbf{X}_\alpha \rightarrow I \rightarrow I \times I$, where that second map is the diagonal map and is thus not surjective. \square

Proof of Theorem 4.3. This follows from Proposition 4.21 and Proposition 4.25. \square

4.3. Proof of Theorem 1.2. Since \mathcal{M} is complete, the right adjoint of G^* exists and can be constructed pointwise (see [B, Thm. 3.7.2] or [M, p. 235]), and Theorem 1.1 implies that $(G^{\text{op}})^*: (\mathcal{M}^{\text{op}})^{\mathcal{D}^{\text{op}}} \rightarrow (\mathcal{M}^{\text{op}})^{\mathcal{C}^{\text{op}}}$ is a right Quillen functor for every model category \mathcal{M}^{op} if and only if G^{op} is fibering (because every model category \mathcal{N} is of the form \mathcal{M}^{op} for $\mathcal{M} = \mathcal{N}^{\text{op}}$).

Proposition 2.27 implies that the functor $G: \mathcal{C} \rightarrow \mathcal{D}$ is cofibering if and only if its opposite $G^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ is fibering, and Theorem 1.1 implies that this is the case if and only if $(G^{\text{op}})^*: (\mathcal{M}^{\text{op}})^{\mathcal{D}^{\text{op}}} \rightarrow (\mathcal{M}^{\text{op}})^{\mathcal{C}^{\text{op}}}$ is a right Quillen functor for every model category \mathcal{M}^{op} , which is the case if and only if $G^*: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ is a left Quillen functor for every model category \mathcal{M} (see Proposition 2.34 and Proposition 2.30).

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